F.Y. B.Sc.(IT) : Sem. I

Discrete Mathematics

Prelim Question Paper Solution

Time : $2\frac{1}{2}$ Hrs.] **[Marks : 75**

Q.1 Attempt the following (any THREE) **[15]**

Q.1(a) Let A = {c, d, f, g}, B = {f, j}, and C = {d, g}. **[5]**

Answer each of the following questions. Give reasons for your answers.

(i) Is $B \subseteq A$?

(ii) Is $C \subseteq A$?

(iii) Is $C \subseteq C$?

(iv) Is C a proper subset of A?

Ans.: Let A = {c, d, f, g}, B = {f, j}, and C = {d, g}.

(i) No, $B \not\subseteq A$ because $j \in B$ but $j \notin A$

(ii) Yes, $C \subseteq A$ as every element of C belongs to A

(iii) Yes, $C \subseteq C$ every element in C is in C

(iv) Yes, $C \subseteq A$ and $A \neq C$ therefore C is a proper subset of A

Q.1(b) Prove the following using truth table **[5]**

(i) $p \rightarrow q = \neg p \wedge q$

(ii) $p \rightarrow q = (\neg p \wedge q) \vee (\neg q \wedge p)$

(iii) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

Ans.: (i) $p \rightarrow q = \neg p \vee q$

p	q	$\neg p$	$p \rightarrow q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

From the truth table it is proved that $p \rightarrow q = \neg p \vee q$

(ii) $p \rightarrow q = (\neg p \wedge q) \vee (\neg q \wedge p)$

p	q	$\neg p$	$\neg q$	$\neg p \vee q$	$\neg q \vee p$	$p \leftrightarrow q$	$(\neg p \vee q) \wedge (\neg q \vee p)$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	T	F	F	F
F	F	T	T	T	T	T	T

From the truth table it is proved that $p \rightarrow q = (\neg p \wedge q) \vee (\neg q \wedge p)$

(iii) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

p	q	r	$p \wedge q$	$p \wedge r$	$q \vee r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	F	F	F
F	T	T	F	F	T	F	F
F	T	F	F	F	T	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

From the truth table it is proved that $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

Q.1(c) Define the following : **[5]**

(i) Universal statement **(ii) Existential universal statement**
(iii) Subset **(iv) Cartesian product**
(v) Relation

Ans.: (i) Universal statement :

The universal quantification of P(x) is the statement
"P(x) for all values of x in the domain"

The notation $\forall x\ P(x)$ denotes the universal quantification of P(x). Here $\forall$ is called the universal quantifier. We read $\forall x\ P(x)$ as "for all x P(x)" or "for every x P(x)".

The statement $\forall x\ P(x)$ is true when P(x) is true for every value of x. The statement $\forall x\ P(x)$ is false when P(x) is false when there is an x for which P(x) is false.

(ii) Existential universal statement:

The existential quantification of P(x) is the proposition
"There exists an element x in the domain such that P(x)."

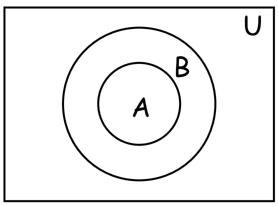
The notation $\exists x\ P(x)$ denotes the existential quantification of P(x). Here $\exists$ is called the existential quantifier.
" The existential quantification $\exists x\ P(x)$ is read as
"There is an x such that P(x),"
"There is at least one x such that P(x),"
or
"For some x P(x)."
The statement $\exists x\ P(x)$ is true when P(x) is true for some value of x. The statement $\exists x\ P(x)$ is false when P(x) is false for all x.

(iii) Subset:

The set A is a subset of set B if and only if every element of A is also an element of B, denoted as $A \subseteq B$

$A \subseteq B$ if $\forall x(x \in A \rightarrow x \in B)$

$A \not\subset B$ denotes that A is not a subset of B, if $\exists x(x \in A \wedge x \notin B)$



Venn diagram showing $A \subseteq B$

For example A = {1,2,3,4,5,6} and B = {2,4,5} Here $B \subseteq A$, However, $A \not\subseteq B$.

(iv) Cartesian product:

Let A and B be the any two given set then Cartesian product or cross product between them is denoted by AxB and is defined as

$$AxB = \{(a,b) / \forall a \in A, \forall b \in B\}$$

Example :

$$A = \{1, 2, 3\}, B = \{a, b\}$$

$$AxB = \{(1,a),(1,b)(2,a),(2,b),(3,a),(3,b\}$$

$$BxA = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$$

$$AxA = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$$

Here if $|A| = m$, $|B| = n$

then $|Axb| = m \cdot n$

$\therefore$ $AxB \neq BxA$ but

$|Axb| = |BxA|$

(v) Relation:

Let A and B be non-empty sets. A relation R from A to B is a subset of the Cartesian product $A \times B$. If $R \subseteq A \times B$ and $(a, b) \in R$, it is said that a is related to b by R, denoted as aRb. If $(a, b) \notin R$ it is denoted as $a \not R b$. For example, A = {1, 2, 3} and B = {a, b}. Then R = {(1, a), (1, b), (2, b), (3, a)} is a relation from A to B.

Q.1(d) (i) Which of the following sets are equal? Justify your answer: **[5]**

$A = \{0, 1, 2\}$ **$B = \{x \in R \mid -1 \le x < 3\}$**

$C = \{x \in R \mid -1 < x < 3\}$ **$D = \{x \in Z \mid -1 < x < 3\}$**

$E = \{x \in Z^+ \mid -1 < x < 3\}$

(ii) Let A = {w, x, y, z} and B = {a, b}. Use the set-roster notation to write each of the following sets, and indicate the number of elements that are in each set :

$A \times B$, $B \times A$, $A \times A$, $B \times B$

Ans.: (i) Given $A = \{0, 1, 2\}$

$B = \{x \in R \mid -1 \le x < 3\} = (-1, 3)$

$C = \{x \in R \mid -1 < x < 3\} = (-1, 3)$

$D = \{x \in Z \mid -1 < x < 3\} = \{0, 1, 2\}$

$E = \{x \in Z^+ \mid -1 < x < 3\} = \{1, 2\}$

Here A = D since it contains same elements.

(ii) Given A = {w, x, y, z} and B = {a, b}

$\therefore$ $A \times B$ = {(w, a), (w, b), (x, a), (x, b), (y, a), (y, b), (z, a), (z, b)}

$\Rightarrow |A \times B| = 8$

$B \times A$ = {(a, w), (a, x), (a, y), (a, z), (b, x), (b, w), (b, y), (b, z)}

$\Rightarrow |B \times A| = 8$

$A \times A$ = {{w, w), (w, x), (w, y), (w, z), (x, w), (x, x), (x, y), (x, z), (y, w), (y, x), (y, y), (y, z), (z, w), (z, x), (z, y), (z, z)}

$\Rightarrow |A \times A| = 16$

$B \times B$ = {(a, a), (a, b), (b, a), (b, b)}

$\Rightarrow |B \times B| = 4$

Q.1(e) Explain Russell's Paradox with an example. **[5]**

Ans.:

- Paradox is a statement or proposition that seems to be self-contradictory but in reality it express a possible truth

  Example:

  (i) You can save money by spending it

  (ii) A rich man is no richer than a poor man

- **Russell's Paradox**

  It is a logical paradox which is stated in terms of set theory. The set of all sets that do not contains themselves as member of that set i.e. the condition for that set is to contain itself is that is should not contain itself.

Q.1(f) Write the following statements using the symbols ~, ∧, ∨ and the indicated letters to represent the component statements h : "Raj is healthy", w : "Raj is wealthy", s : "Raj is wise" **[5]**

(i) Raj is healthy and wealthy but not wise.
(ii) Raj is not wealthy but he is healthy and wise.
(iii) Raj is neither healthy, wealthy nor wise.
(iv) Raj is neither wealthy, nor wise but he is healthy.
(v) Raj is wealthy but he is not both healthy and wise.

Ans.: (i) Raj is healthy and wealthy but not wise.
$h \wedge w \wedge \sim s$
(ii) Raj is not wealthy but he is healthy and wise.
$\sim w \wedge h \wedge s$
(iii) Raj is neither healthy, wealthy nor wise.
$\sim h \wedge \sim w \wedge \sim s$
(iv) Raj is neither wealthy, nor wise but he is healthy.
$\sim w \wedge \sim s \wedge h$
(v) Raj is wealthy but he is not both healthy and wise.
$w \wedge \sim (h \wedge s)$

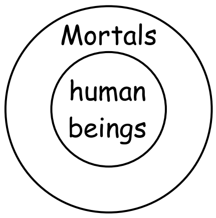
Q.2 Attempt the following (any THREE) **[15]**

Q.2(a) Indicate whether the following arguments are valid or invalid. Support your answer with diagrams : **[5]**

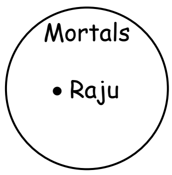
(i) All human beings are mortal.
Raju is mortal.
∴ Raju is human being.

(ii) All polynomial functions are differentiable.
All differentiable functions are continuous.
∴ All polynomial functions are continuous.

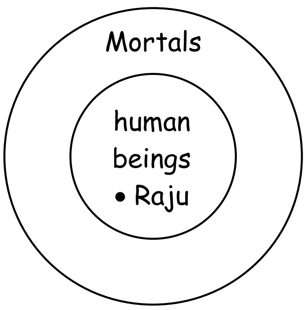
Ans.: (i) The argument is invalid.
The major premises "All human beings are mortal" is represented diagrammatically by



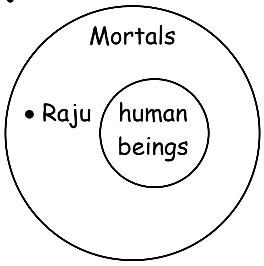
The minor premises "Raju is mortal" is presented diagrammatically by



Raju is mortal indicates two cases as shown in the diagram
Case I : Raju is a human being.

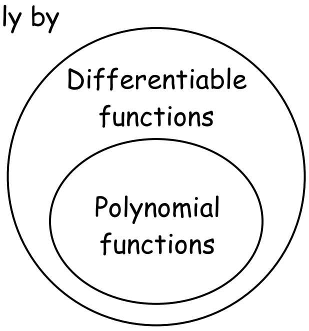


Case II : Raju is mortal but not a human being.

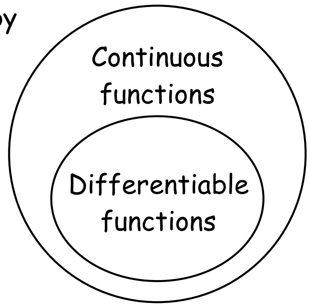


Because the conclusion does not follow from the premises, the argument is invalid.

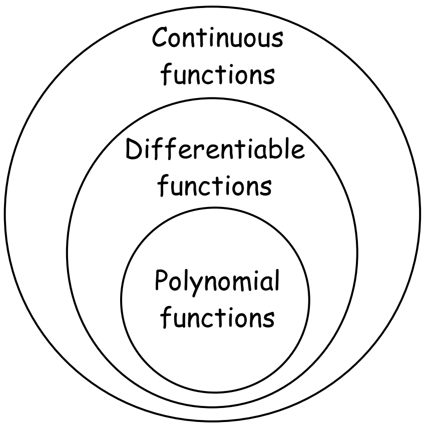
(ii) The argument is valid.
The premises "All polynomial functions are differentiable" is represented diagrammatically by



The premises "All differentiable functions are continuous" is represented diagrammatically by



From the two premises we can conclude that "all polynomial functions are continuous."



$\therefore$ The given argument is valid.

Q.2(b) Define prime numbers and composite numbers. Express the definition using symbols. Prove that every integer greater than 1 is either prime or composite. Write first six prime numbers and composite numbers. **[5]**

Ans.: An integer n is **prime** if, and only if, $n > 1$ and for all positive integers r and s, if $n = r \cdot s$, then $r = 1$ or $s = 1$.

An integer n is **composite** if, and only if, $n > 1$ and $n = r \cdot s$ for some positive integers r and s with $r \neq 1$ and $s \neq 1$.

Symbolically, if n is an integer and n > 1, then
n is prime ⇔ ∀ positive integers r and s, if n = r · s then r = 1 or s = 1

n is composite ⇔ ∃ positive integers r and s such that n = r · s and r ≠ 1 and s ≠ 1.
For any integer greater than 1, it is either prime or composite. The definitions of prime and composite numbers are negations of each other.
For all integers n > 1, n is prime
$\forall(r, s)\ (n = r \cdot s \rightarrow (r = 1 \vee s = 1))$

Taking negation gives: n is not prime
$\sim (\forall(r, s)\ (n = r \cdot s \rightarrow (r = 1 \vee s = 1)))$
$\equiv \exists(r, s) \sim(\sim(n = r \cdot s) \vee (r = 1 \vee s = 1))$
$\equiv \exists(r, s)\ (n = r \cdot s) \wedge (r \neq 1 \wedge s \neq 1))$

The negation is same as the definition of composite number hence if the integer greater than 1 is either prime or composite.
First six prime numbers: 2, 3, 5, 7, 11, 13
First six composite numbers: 4, 6, 8, 9, 10, 12

Q.2(c) State the Euclidian algorithm. Find the gcd of (330,156) by using Euclidean algorithm. **[5]**

Ans.: Euclidian Algo
Input : m and n as given no.
Step 1 : r = n
Step 2: While (n≠0)
r = m mod n
m = n
n = r
end while
gcd = m
output : gcd.

Here m = 330, n = 156

step 1: r = 156
step 2 : while (156 = n ≠ 0)

Condition	T/F	Interrelation no.	r = m mod n	m = n	n = R
156 ≠ 0	T	1	r = 330 mod 156 r = 18	m = 156	n = 18
18 ≠ 0	T	2	r = 156 mod 18 = 12	m = 18	n = 12
12 ≠ 0	T	3	r = 18 mod 12 = 6	m = 12	n = 6
6 ≠ 0	T	4	r = 12 mod 6 = 0	m = 6	r = 0
0 ≠ 0	F				

g cd (330, 156) = m = 6

Q.2(d) Let D = {-48, -14, -8, 0, 1, 3, 16, 23, 26, 32, 36}. **[5]**

Determine which of the following statements are true and which are false. Provide counterexamples for those statements that are false.

(i) $\forall x \in D$, if x is odd then x >0.

(ii) $\forall x \in D$, if x is less than 0 then x is even.

(iii) $\forall x \in D$, if x is even then x < 0.

(iv) $\forall x \in D$, if the ones digit of x is 2, then the tens digit is 3 or 4,

(v) $\forall x \in D$, if the ones digit of x is 6, then the tens digit is 1 or 2.

Ans.: **(i) True.** All the odd numbers in D are positive.

(ii) True. All the negative numbers are even in D.

(iii) False. There are even numbers 16, 26, 32, 36 greater than 0 belonging to D.

(iv) True. $32 \in D$ hence is true and $42 \notin D$ hence is false. True or False is True. The given statement is true.

(v) False. x = 36 is a counterexample because the ones digit of x is 6 and the tens digit is neither 1 nor 2.

Q.2(e) Let D be the set of all students at your school, and let M(s) be "s is a math major," let C(s) be "s is a computer science student," and let E(s) be "s is an engineering student." **[5]**

Express each of the following statements using quantifiers, variables, and the predicates M(s), C(s), E(s).

a. There is an engineering student who is a math major.

b. Every computer science student is an engineering student.

c. No computer science students are engineering students.

d. Some computer science students are also math majors.

e. Some computer science students are engineering students are some are not.

Ans.:
1. $\exists s\ E(s) \rightarrow M(s)$
2. $\forall s\ C(s) \rightarrow E(s)$
3. $\forall s \sim C(s) \rightarrow E(s)$
4. $\exists s,\ C(s) \rightarrow M(s)$
5. $\exists s, t\ (C(s) \rightarrow E(s) \wedge (C(t) \rightarrow N\ E(t))$

Q.2(f) Prove that $\sqrt{5}$ is irrational. **[5]**

Ans.: Proof by contradiction, suppose that $\sqrt{5}$ is national.

By definition of national

$\sqrt{5} = \frac{a}{b}$ for some integers a a & b, $b \neq 0$.

Assume that a & b have no common factors. Squaring both sides if the equation we gets,

$5 = \frac{a^2}{b^2}$

$\therefore\ 5b^2 = a^2$ ...(1)

Thus a^2 is divisible by 5 so a is also divisible by 5.

$\therefore$ By definition of divisibility then a = 5k for some integer k &

so $a^2 = 25k^2$ ...(2)

Substituting equation (2) in equation (1) gives $5b^2 = 25k^2$ dividing both sides by 5 we get,

$b^2 = 5k^2$

hence, b^2 is divisible by 5.

$\therefore$ b is also divisible by 5.

Consequently, both a & b are divisible by 5 which contradicts the assumption that a & b have no common factor. Thus the supposition is false.

So, $\sqrt{5}$ is irrational.

Q.3 Attempt the following (any THREE) : [15]

Q.3(a) Define g: Z → 2 by the rule g(n) = 4n – 5, for all integers n. [5]

(i) Is g one-to-one? Prove or give a counterexample.

(ii) Is g onto? Prove or give a counterexample.

Ans.: $g : Z \to Z$ defined as $g(b) = 4n - 5$

(i) Let n_1 and n_2 be integers such that $g(n_1) = g(n_2)$.

$\therefore\ 4n_1 - 5 = 4n_2 - 5$

$\therefore\ 4n_1 = 4n_2$

$\therefore\ n_1 = n_2$

$\because\ n_1 = n_2$ for $g(n_1) = g(n_2)$, g is one-to-one.

(ii) Let $y = g(n)$

$\therefore\ y = 4n - 5$

$n = \dfrac{y+5}{4}$

Let y = 0, then $n = \dfrac{5}{4} = 1.25 \notin Z$

$\therefore$ g(n) is not onto.

Q.3(b) Prove that $n^3 - 7n + 3$ is divisible by 3, for each integer $n \geq 0$. [5]

Ans.: Let P(n) be the proposition.

P(n): $n^3 - 7n + 3$ is divisible by 3.

Step 1: To show that P(3) is true.

To show that $-7 \times (1) + 3$ is divisible by 3

i.e. $-7 + 3$ is divisible by 3

i.e. -3 is divisible by 3

$\therefore$ P(0) is true.

Step 2: Show that for all integers $k \geq 0$ if P(k) is true then P(k + 1) is also true.

Suppose k is any integer $k \geq 0$, such that $k^3 - 7k + 3$ is divisible by 3

i.e. $k^3 - 7k + 3 = 3m$ where $m \in z$ ...(1)

To show that P(k + 1) is true.

i.e. To show that $(k^3 + 1)^3 - 7(k + 1) + 3$ is divisible by 3.

$(k + 1)^3 - 7(k + 1) + 3 = k^3 + 3k^2 + 3k + 1 - 7k - 7 \pm 3$

$= (k^3 - 7k + 3) + 8k^2 + 3k - 6$

$= 3m + 3(k^2 + k - 2)$ (from equation (1))

$= 3m + 3n \quad = 3(m + n)$

$= 3t$

where t is any integer being addition of integers.

$\therefore$ P(k + 1) is true.

i.e. P(k + 1) is divisible by 3

$\therefore$ By principle of Mathematical Induction, $n^3 - 7n + 3$ is divisible by 3

Q.3(c) Define : (i) Function (ii) Logarithm (iii) Logarithmic funciton (iv) Boolean function (v) Image and Inverse Image [5]

Ans.: **(i) Function :**

Let A and B be the any two given set the subset f of AxB [$f \in A \times B$] is said to be a function from A to B if $f\ \forall x \in A\ \exists$ unique $y \in B$ i.e.

$f = \{(x, y)\}$

$\forall x \in A, \forall y \in B$ and $\forall x \in A\ \exists$ unique y + B

$f : A \to B$

(ii) Logarithm:

Let b be a positive real number. For each positive real number x, the logarithm with base b of x, written $\log_b x$, is the exponent to which b must be raised to obtain x. It is written as $\log_b x = y \Leftrightarrow b^y = x$.

(iii) Logarithmic function:

The logarithmic function with base b is the function from R^+ to R that takes each positive real number x to $\log_b x$.

(iv) Boolean function:

An (n-place) Boolean function f is a function whose domain is the set of all ordered n-tuples of 0's and l's and whose co-domain is the set {0, 1}. More formally, the domain of a Boolean function can be described as the Cartesian product of n copies of the set {0, 1}, which is denoted $\{0, l\}^n$. Thus f: $\{0,1\}^n \rightarrow \{0\ 1\}$.

(v) Image and Inverse Image:

Let f be a function from X to Y. The set of all values of f taken together is called the range of f or the image of X under f. Symbolically, Range of f = Image of X under f = $\{y \in Y \mid y = f(x)$, for some x in X}.

Given an element y in Y, there may exist elements in X with y as their image. If f (x) = y, then x is called a preimage of y or an inverse image of y. The set of all inverse images of y is called the inverse image of y. Symbolically, Inverse Image of y = $\{x \in X \mid f(x) = y\}$.

Q.3(d) Define surjective function and inverse function. Find the inverse of the following functions : [5]

(i) Define f : $Z \rightarrow Z$ by the rule f(n) = 2n for all integers n.

(ii) Define G: $R \rightarrow R$ by the fule G(x) = 4x − 5 for all real numbers x.

Ans.: Surjective function: Let F be a function from a set X to a set Y. F is onto (or Surjective) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that y = F(x).

Symbolically:

F: $X \rightarrow Y$ is onto $\Leftrightarrow \forall y \in Y, \exists x \in X$ such that F(x) = y.

In other words, if Range of f = Y, the function is an Surjective function.

Inverse Function: Suppose F: $X \rightarrow Y$ is a one-to-one correspondence; that is, suppose F is one-to-one and onto. Then there is a function F^{-1}: $Y \rightarrow X$ that is defined as follows:

Given any element y in Y,

$F^{-1}(y)$ = that unique element x in X such that F(x) equals y.

In other words, $F^{-1}(y) = x \Leftrightarrow y = F(x)$.

(i) Let y = 2n

$\therefore$ n = y/2

Hence, $f^{-1}(y) = y/2$.

(ii) Let y = 4x – 5

$\therefore$ x = (y + 5)/4

Hence $G^{-1}(y) = (y + 5)/4$.

Q.3(e) Let f: $Z \rightarrow Z$ be the successor function and let g: $Z \rightarrow Z$ be the squaring function. Then f(n) = n + 1 for all $n \in Z$ and g(n) = n^2 for all $n \in Z$. [5]

(i) Find the compositions g*f and f*g.

(ii) b . Is g*f = f*g? Explain

Ans.: (i) The functions g o f and f o g are defined as follows:

$(g \circ f)(n) = g(f(n)) = g(n + 1) = (n + 1)^2$ for all $n \in Z$,

and

$(f \circ g)(n) = f(g(n)) = f(n^2) = n^2 + 1$ for all $n \in Z$.

Thus

$(g \circ f)(n) = (n + 1)^2$ and $(f \circ g)(n) = n^2 + 1$ for all $n \in Z$.

(ii) Two functions from one set to another are equal if, and only if, they take the same values. In this case,

$(g \circ f)(1) = (1 + 1)^2 = 4$, whereas $(f \circ g)(1) = 1^2 + 1 = 2$.

Thus the two functions g o f and f o g are not equal: gof ≠ fog.

Q.3(f) Find the first four terms of each of the recursively defined sequence **[5]**
$S_k = S_{k-1} + 2S_{k-2}$, for all integers k ≥ 2 $S_0 = 1$, $S_1 = 1$

Ans.: Here, $S_0 = 1$, $S_1 = 1$

$S_2 = S_1 + 2S_0 \quad = 1 + 2 \times 1 \quad = 1 + 2$

$\therefore \; S_2 = 3$

$S_3 = S_2 + 2S_1 \quad = 3 + 2 \times 1 \quad = 3 + 2$

$\therefore \; S_3 = 5$

$S_4 = S_4 - 1 + 2S_{4-2} \quad = S_3 + 2\,S_2 \quad = 5 + 2(3)$

$\therefore \; S_4 = 11$

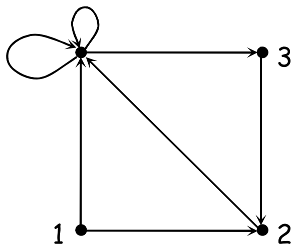
$S_5 = S_{5-1} + 2\,S_{5-2} \quad = 11 + 2(5)$

$\therefore \; S_5 = 21$

Q.4 Attempt the following (any THREE) **[15]**

Q.4(a) Let S = {(0, 0), (0, 3), (1, 0), (1, 2), (2, 0), (3, 2)}. Find S_t, the transitive closure of S. **[5]**

Ans.:



S_t = {(0, 0), (0, 2), (0, 3), (1, 0), (1, 2), (1, 3),
(2, 0), (2, 2), (2, 3), (3, 0), (3, 2), (3, 3)}

Q.4(b) (i) If R and S are reflexive, is R ∩ S reflexive ? Why? **[5]**
(ii) If R and S are symmetric, is R ∩ S symmetric ? Why?
(iii) If R and S are transitive, is R n S transitive? Why?

Ans.: (i) Let Δ = {(a, a) | a ∈ A}, be a relation on the set A.

∵ R is reflexive, Δ ⊆ R
∵ S is reflexive, Δ ⊆ S
∵ Δ ⊆ R and Δ ⊆ S, Δ ⊆ R ∩ S
Hence R ∩ S is reflexive.

(ii) Suppose (x, y) ∈ R ∩ S
By definition of intersection (x, y) ∈ R and (x, y) ∈ S.
∵ R is symmetric (x, y) ∈ R means (y, x) ∈ R.
Similarly, since S is symmetric (x, y) ∈ S means (y, x) ∈ S.
∴ (y, x) ∈ R ∩ S if (x, y) ∈ R ∩ S
Hence R ∩ S is symmetric.

(iii) Suppose (x, y) ∈ R ∩ S and (y, z) ∈ R ∩ S
By definition of intersection, (x, y) ∈ R and (y, z) ∈ R
also (x, y) ∈ S and (y, z) ∈ S
∵ R and S are transitive it follows that (x, z) ∈ R and (x, z) ∈ S
∵ (x, z) ∈ R ∩ S
Hence R ∩ S is transitive if R and S are transitive.

Q.4(c) Define the following: **[5]**
(i) Trail **(ii) Connected Graph** **(iii) Spanning Tree**
(iv) Hamiltonian Graph **(v) Hamiltonian Cycle**

Ans.: (i) Trail: A trail from v to w is a walk from v to w that does not contain a repeated edge.

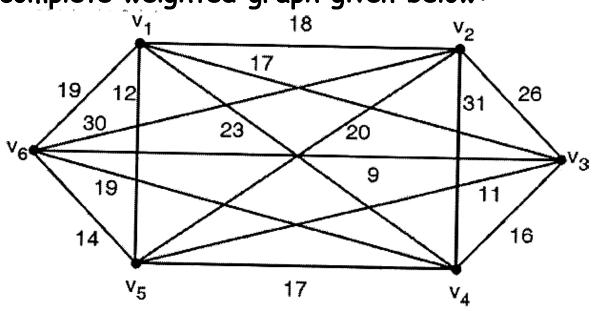
(ii) Connected Graph: The graph G is connected if, and only if, given any two vertices v and w in G, there is a walk from v to w. Symbolically, G is connected $\Leftrightarrow$ $\forall$ vertices v, w $\in$ V(G), $\exists$ a walk from v to w.

(iii) Spanning Tree: A spanning tree for a graph G is a subgraph of G that contains every vertex of G and is a tree.

(iv) Hamiltonian Graph: A Hamiltonian graph is a graph possessing a Hamiltonian cycle.

(v) Hamiltonian Cycle: Given a graph G, a Hamiltonian cycle for G is a simple cycle that includes every vertex of G. That is, a Hamiltonian cycle for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, except for the first and the last, which are the same.

Q.4(d) Find Shortest Path from vertex v1 to all the vertices by applying Dijkshtra's Algorithm for the complete weighted graph given below: **[5]**



Ans.: Initially the paths are :

V_1	V_2	V_3	V_4	V_5	V_6
0	∞	∞	∞	∞	∞

Shortest path from V_1 to its adjacent vertices :

V_1	V_2	V_3	V_4	V_5	V_6
0	18	17	23	12	19

{V_1, V_5} are fixed shortest path from $V_1 \rightarrow V_5$ to its adjacent vertices.

V_1	V_2	V_3	V_4	V_5	V_6
0	18	17	23	12	19

{V_1, V_5, V_3} are fixed shortest path from V_3 to its adjacent vertices.

V_1	V_2	V_3	V_4	V_5	V_6
0	18	17	23	12	19

{V_1, V_5, V_3} are fixed shortest path from V_2 to its adjacent vertices.

V_1	V_2	V_3	V_4	V_5	V_6
0	18	17	23	12	19

Q.4(e) A relation R from R to R as follows : **[5]**
For all (x, y) $\in$ R $\times$ R, x R y $\Leftrightarrow$ y = 2|x|
Draw the graphs of R and R^{-1} in the Cartesian plane. Is R^{-1} a function?

Ans.: Given that Relation R is from R to R defined as :
for all (x, y) $\in$ R x R, x R y $\Leftrightarrow$ y = 2|x|
A point (u, v) is on the graph of R^{-1} if and only if (u, v) is on the graph of R.

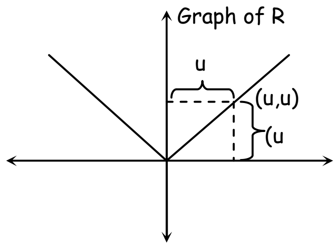
Note that, if $x \geq 0$ then the graph of $y = 2|x|$
$= 2x$ is a straight line with slop 2

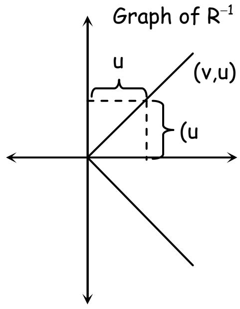
And if $x < 0$ then the graph of $y = 2\,|x|$
$= x\,2(-x)$
$= -2x$ is a straight line with slop – 2

$R = \{(x, y) \mid y = 2|x|$ $R^{-1} = \{(y, x) \mid y = 2\,|x|\}$

x	y
0	0
1	2
–1	2
2	4
–2	4

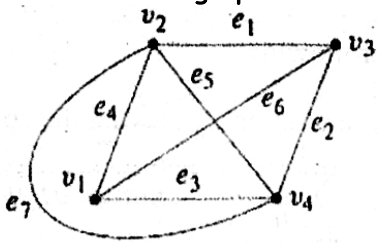
y	X
0	0
2	1
4	2
4	–2





R^{-1} is not a function,
Since both (2, 1) and (2, –1) are in R^{-1}.

Q.4(f) Show that the graph below does not have an Euler circuit. [5]



Ans.: In above graph vertices v_1 and v_3 both have degree 3 which is odd.
According to theorem if a graph has an Euler circuit then every vertex of the graph has positive even degree which is not true in this case.
Hence this graph does not have an Euler circuit.

Q.5 Attempt the following (any THREE) [15]

Q.5(a) (i) If any seven digits could be used to form a telephone number, how many seven-digit telephone numbers would not have any repeated digits? [5]

(ii) How many seven-digit telephone numbers would have at least one repeated digit?

(iii) What is the probability that a randomly chosen seven-digit telephone number would have at least one repeated digit?

Ans.: (i) Seven digit telephone number without repeating any digits :
1st digit can be chosen in 10 ways
2nd digit can be chosen in 9 ways, and so on
7th digit can be chosen in 4 ways
$\therefore$ Total number of ways $= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4$
$= 604800$

(ii) Number of phone numbers with atleast one repeated digit
= total number of phone numbers – number of phone numbers with no repeated digits
$= 10^7 - 604800 \quad = 9395200$

(iii) The probability that a randomly chosen seven digit telephone number would have at least one repeated digit is

No. of phone number with at least one

$$= \frac{\text{repeated digit}}{\text{total number of phone numbers}} = \frac{9395200}{10^7}$$

$\approx 93.95\%$

Q.5(b) Suppose a person offers to play a game with you. In this game, when you draw a card from a standard 52-card deck, if the card is a face card you win Rs. 3 and if the card is anything else you lose Re. 1. If you agree to play the game, what is your expected gain or loss? **[5]**

Ans.: As there are 12 face cards and 40 non-face cards. The probability of drawing a face card is 12/52. The probability of drawing a non-face card is 40/52.

$\therefore$ The expected gain or loss = $\frac{12}{52}\times 3+\frac{40}{52}\times(-1)=-\frac{4}{52}=-0.077$

There would be an expected loss of 7.7 paise.

Q.5(c) (i) How many ways can the letters of the word QUICK be arranged in a row? **[5]**

(ii) How many ways can the letters of the word QUICK be arranged in a row if the Q and the U must remain next to each other in the order QU?

(iii) How many ways can the letters of the word QUICK be arranged in a row if the letters QU must remain together but may be in either the order QU or the order UQ?

Ans.: (i) The answer is the number of permutations of the five letters in QUICK which equals 5! = 120

(ii) Because QU (in order) is to be considered as a single unit the answer is the number of permutations of the four symbols QU, I, C, K (i.e.) 4! = 24

(iii) By part (ii) there are 4! arrangements of UQ, I, C, K

$\therefore$ By addition rule,

4! + 4! = 24 + 24

= 48 arrangements is all

Q.5(d) A coin is loaded so that the probability of heads is 0.6. Suppose the coin is tossed ten times. **[5]**

(i) What is the probability of obtaining eight heads?

(ii) What is the probability of obtaining at least eight heads?

Ans.: Given that probability of heads is 0.6 let probability of heads be P

(i.e. probability of success)

$\therefore$ Probability of failure q = (1 − p) = 0.4

coin is tossed 10 times

$\therefore$ Here, n = 10, p = 0.6, q = (1 − p)

The probability of exactly x success is a binomial experiment B (n, p) is given by B(n, p) = $\binom{n}{x} p^x q^{n-x}$

(i) P(eight heads) = P(x = 8) $= \binom{10}{8}(0.6)^8(0.4)^2 = 0.1209$

(ii) P(Atleast eight heads) = $P(x \geq 8)$

= P(x = 8) + P(x = 9) + P(x = 10)

$= \binom{10}{8}(0.6)^8(0.4)^2 + \binom{10}{9}(0.6)^9(0.4)^1 + \binom{10}{9}(0.6)^{10}(0.4)^0$

= 0.167

Q.5(e) Let A = {1, 2, 3, 4, 5., 6, 7, 8} **[5]**

(i) If five integers are selected from A must at least one pair of the integers have a sum of 9? Explain

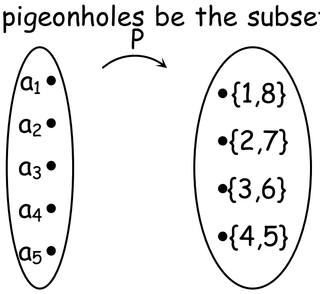
(ii) If four integers are selected from A, must at least one pair of the integers have a sum of 9? Explain.

Ans.: Given : A = {1, 2, 3, 4, 5, 6, 7, 8}

(i) If five integers are selected from A then atleast one pair of integers will have sum of 9.

Explanation : Partition the set A into the following four disjoint subsets {1, 8}, {2, 7}, {3, 6} and {4, 5}. Observe that each of the integers in A occurs in exactly one of the four subsets and that the sum of the integers in each subset is 9. Thus if five integers from A are chosen then by the pigeonhole principle two must be from the same subset. It follows that the sum of these two integers is 9.

To see precisely how the pigeonhole principle applies, let the pigeons be the five selected integers (call them a, a_2, a_3, a_4, and a_5) and let the pigeonholes be the subsets of the partition.



The function P from pigeons to pigeonholes is defined by letting $P(a_i)$ be the subset that contains a_i.

Because there are more pigeons than pigeonholes atleast two pigeons must go to the same hole.

$\Rightarrow$ Atleast one pair of integers have a sum of 9

(ii) If four integers are selected from A it is not necessary to get sum of 9 from one pair. Counter example : 1, 2, 3, 4 is selected. No two integers sum up to 9.

Q.5(f) How many positive three-digit integers are multiples of 6? What is the probability that a randomly chosen positive three-digit integer is a multiple of 6? What is the probability that a randomly chosen positive three-digit integer is a multiple of 7? **[5]**

Ans.: (i) The diagram shows that there are many positive three digit integers that are multiple of 6 as there are integers from 16 to 166 inclusive. By theorem of inclusion exclusion there are 166 – 17 + 1 or 150 such integers.

(ii) There are 999 – 100 + 1 = 990 positive three digit integers in all & by part (i), 150 of there are multiple of 6. So the probability that a randomly chosen positive three digit integer is a multiple of 6 is

150/900 = 1/6 = 16.667

(iii) There are 900 positive three digit integers out of which 142 – 15 + 1 = 128 such integers which are multiple of 7, hence the probability that a randomly chosen positive three digit integers is multiple of 7 is

128/900 = 32/225 = 14.222%

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