Applied Mathematics

Prelim Question Paper Solution

Time : $2\frac{1}{2}$ Hrs.] **[Marks : 75**

Q.1 Attempt the following (any THREE) **[15]**

Q.1(a) Verify Caley-Hamilton theorem for the given matrix, also find inverse if exists. **[5]**

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Ans.: Let; $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$|A - \lambda I| = \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

Replace $\lambda = A$

$\therefore A^3 - 6A^2 + 9A - 4 = 0$

$$A^2 = A \times A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -6 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ Caylay Hamilton Theorem is verified

Now;

$A^3 - 6A^2 + 9A - 4I = 0$

Pre Multiply above by A^{-1}

$\therefore A^{-1} A \cdot A^2 - 6A^{-1} \cdot A \cdot A + 9A^{-1} \cdot A - 4A^{-1} I = 0$

$A^2 - 6A + 9I - 4A^{-1} = 0$

$\therefore 4A^{-1} = A^2 - 6A + 9I$

$\therefore A^{-1} = \frac{1}{4}\left[A^2 - 6A + 9I\right]$

$$= \frac{1}{4}\left\{\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ -5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Q.1(b) For different values of k, discuss the following equations: **[5]**
$x + 2y - z = 0;\ 3x + (k + 7)y - 3z = 0;\ 2x + 4y + (k - 3)z = 0$

Ans.: **Step 1:** $x + 2y - z = 0;\ 3x + (k + 7)y - 3z = 0;\ 2x + 4y + (k-3)z = 0$

It can be written as;

$Ax = z$

Where $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & k+7 & -3 \\ 2 & 4 & k-3 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $z = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Step 2: $[A|z] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & k+7 & -3 & 0 \\ 3 & 4 & k-3 & 0 \end{array}\right]$

$R_2 - 3R_1$ & $R_3 - 2R_1$

$$= \left[\begin{array}{cc|cc} 1 & 2 & -1 & 0 \\ 0 & k+1 & 0 & 0 \\ 0 & 0 & k-1 & 0 \end{array}\right] \qquad \ldots(1)$$

Step 3: The system have infinite number of solutions if r = n i.e. P(A) < 3 (number of unknown = n = 3) From the last matrix it is possible only if

$k + 1 = 0$ or $k - 1 = 0$ i.e. $k = -1$ <u>OR</u> $k = 1$

For k = 1 :

$$[A|z] = \left[\begin{array}{ccc|c} 1 & 2 & - & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \ldots \text{From (1)}$$

$\therefore\ x + 2y - z = 0\ ;\ 2y = 0$

$\therefore\ y = 0$

Put $z = k_1$; $-x = z = k_1$ i.e. $x = k_1\ ;\ y = 0;\ z = k_1$

For k = –1 :

$$[A|z] = \left[\begin{array}{ccc|c} 1 & 2 & - & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \qquad \ldots \text{From (1)}$$

$x + 2y - z = 0;\ \ -2z = 0$

$\therefore\ z = 0;\ \ x = -2y;$ Put $y = k_2$

Put $z = k_1$ $\therefore\ x = -2k_2$ i.e. $x = -2k_2\ ;\ y = k_2$ & $z = 0$

Q.1(c) Find the Characteristic values and characteristic vectors of the given matrix. **[5]**

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Ans.: Characteristic Equation

$A - \lambda I = \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$

$= (8 - \lambda)[(7 - \lambda)(3 - \lambda) - 16] + 6[- 6(3 - \lambda) + 8] + 2[24 - 2(7- \lambda)]$

$= -\lambda^3 + 18\lambda^2 - 45\lambda = 0$

$= \lambda^3 - 18\lambda^2 + 45\lambda = 0 \quad \rightarrow (1)$

Replace λ by "A" to get characteristics equation

$\therefore\ A^3 - 18A^2 + 45A = 0$

Roots of "λ" from equation (1) is λ =0, 3, 15

(i) For $\lambda = 0$

$$A - \lambda I = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$8x_1 - 6x_2 + 2x_3 = 0$
$-6x_1 + 7x_2 - 4x_3 = 0$
$2x_1 - 4x_2 + 3x_3 = 0$

$\therefore \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$

$\therefore x_1 = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}^T$

(ii) For $\lambda = 3$

$$A - \lambda I = \begin{bmatrix} 5 & -6 & 2 \\ -6 & 2 & -4 \\ 2 & -4 & -2 \end{bmatrix}$$

$5x_1 - 6x_2 + 2x_3 = 0$
$-6x_1 + 2x_2 - 4x_3 \ 0$
$2x_1 - 4x_2 - 2x_3 = 0$

$\therefore \frac{x_1}{-2} = \frac{x_2}{-1} = \frac{x_3}{2}$

$\therefore x_2 = \begin{bmatrix} -2 & -1 & 2 \end{bmatrix}^T$

(iii) For $\lambda = 15$

$$A - \lambda I = \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}$$

$-7x_1 - 6x_2 + 2x_3 = 0$
$-6x_1 - 8x_2 - 4x_3 = 0$
$2x_1 - 4x_2 - 12x_3 = 0$

$\therefore \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$

$\therefore x_3 = \begin{bmatrix} 2 & -2 & 1 \end{bmatrix}^T$

Q.1(d) Express $\frac{-1}{2} + \frac{\sqrt{3}}{2} i$ in polar form. **[5]**

Ans.: Consider, $z = \frac{-1}{2} + \frac{\sqrt{3}}{2} i$

Compare, $z = x + iy$

$\therefore \quad x = \frac{-1}{2}$ & $y = \frac{\sqrt{3}}{2}$

Polar form of $z = r[\cos\theta + i\sin\theta]$ … (1)

Where "r" is modulus & θ is amplitude of z.

Now, $|z| = r = \sqrt{x^2 + y^2}$

$\therefore \quad r = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}}$

$\therefore \quad r = 1$

Amplitude of z :

$x = \frac{-1}{2}, y = \frac{\sqrt{3}}{2}$ $\quad x < 0$ & $y > 0$

$\therefore$ It is in 2^{nd} quadrant

We know ;

$z = 0 = \pi - \tan^{-1}|y/x|$

$$\therefore \quad \theta = \pi - \tan^{-1}\left|\frac{\sqrt{3}/2}{-1/2}\right| = \pi - \tan^{-1}(\sqrt{3}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Substitute r & θ in equation (1)

$$\therefore \quad z = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$$

Q.1(e) Prove that $(1 + i\sqrt{3})^8 + (1 - i\sqrt{3})^8 = -2^8$ **[5]**

Ans.: Let ; $z_1 = 1 + i\sqrt{3}$, $= r[\cos\theta + i\sin\theta]$ … (1)

Where; r is modules of z & θ is argument.

Modulus of z :

$$|z| = r\sqrt{x^2+y^2} = \sqrt{(1)^2+(\sqrt{3})^2} = \sqrt{1+3} = 2$$

Amplitude of z :

$x > 0$ & $y > 0$ $\quad\therefore$ It is in 1^{st} quadrant

$\therefore \quad \theta = \tan^{-1}|y/x| = \tan^{-1}\pi/3$

Substitute r & θ in equation (1)

$$\therefore \quad z = 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$$

by when $z_2 = 1 - i\sqrt{3}$

then $z_2 = 2\left[\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right]$

Now, $z_1^8 + z_2^8 = (1 + i\sqrt{3})^8 + (1 - i\sqrt{3})^8$

$$= \left\{2\left[\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\right]\right\}^8 + \left\{2\left[\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right]\right\}^8$$

By De–Moivre's Theorem,

$$= 2^8\left[\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right] + 2^8\left[\cos\frac{8\pi}{3} - i\sin\frac{8\pi}{3}\right]$$

$$= 2^8\left[\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3} + \cos\frac{8\pi}{3} - i\sin\frac{8\pi}{3}\right]$$

$$= 2^8\left[2.\cos\left(\frac{8\pi}{3}\right)\right] = 2^9\cos\left(3\pi - \frac{\pi}{3}\right)$$

$$= 2^9\left(-\cos\frac{\pi}{3}\right) = -2^9 x \frac{1}{2} = -2^8$$

$\therefore \quad (1 + i\sqrt{3})^8 + (1 - i\sqrt{3})^8 = -2^8$

Q.1(f) Expand $(1 + \cos x + i\sin x)^n$ **[5]**

Ans.: $= (2\cos^2 x/2 + 2\sin x/2 \cdot \cos x/2)^n$

$= 2^n\cos^n x/2\,(\cos x/2 - i\sin x/2)^n$

$= 2^n\cos^n x/2\left(\cos\frac{nx}{2} - i\sin\frac{nx}{2}\right) \rightarrow$ By D.T.

Q.2 Attempt the following (any THREE) **[15]**

Q.2(a) Solve the Differential Equation dy / dx + $x^2y = x^5$ **[5]**

Ans.: $\frac{dy}{dx} + x^2y = x^5$

Above equation is in Linear Equation form

i.e. $\frac{dy}{dx} + Py = Q$ $\quad P = x^2, \ \theta = x^5$

$I.F = e^{\int P\,dx} = e^{\int x^2 dx} = e^{x^3/3}$

Solution:

$y \times I.F = \int Q \times I.F\,dx$

$y \times e^{\frac{x^3}{3}} = \int x^5 \cdot e^{\frac{x^3}{3}} dx$

Let $\frac{x^3}{3} = t \therefore x^3 = 3t \qquad \therefore x^3 dx = dt$

$y \times e^{x^3/3} = \int 3t\,e^t\,dt = 3\,[te^t - e^t] + c = 3\left[\frac{x^3}{3}e^{x^3/3} - e^{x^3/3}\right] + c$

$y \times e^{x^3/3} = [x^3 e^{x^3/3} - 3e^{x^3/3}] + c$

Q.2(b) Solve the following Equation $x^2p^2 - 2xpy + (2y^2 - x^2) = 0$ **[5]**

Ans.: $x^2p^2 - 2xpy + (2y^2 - x^2) = 0$

Solvable by "P"

Compare with $ap^2 + bp + c$

$a = x^2;\ \theta = -2xy;\ c = 2y^2 - x^2$

$P = \frac{2xy + \sqrt{4x^2y^2 - 4x^2(2y^2 - x^2)}}{2x^2} = \frac{2xy \pm \sqrt{4x^4 - 4x^2y^2}}{2x^2}$

$P = \frac{y}{x} \pm \sqrt{1 - \frac{y^2}{x^2}}$

$\frac{dy}{dx} = \frac{y}{x} \pm \sqrt{1 - \frac{y^2}{x^2}}$

Let $\frac{y}{x} = V \quad \therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = v \pm \sqrt{1 - v^2}$

$\frac{dv}{\sqrt{1 - v^2}} = \pm\frac{dx}{x}$

Integrating on both the side

$\int \frac{dv}{1 - v^2} = \pm\int \frac{dx}{x}$

$\sin^{-1} v = \pm \log x + c$

$\sin^{-1}\left(\frac{y}{x}\right) = \pm \log x + c$

Q.2(c) Solve $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \cos 2x$ **[5]**

Ans.: The given DE is $\quad (D^3 + D^2 - D - 1)\,y = \cos 2x$

Step 1: C.F (y_c)

y_c = A.E is $D^3 + D^2 - D - 1 = 0$

$\therefore \ D^2(D + 1) - 1(D + 1) = 0 \quad \Rightarrow \quad (D^2 - 1)(D + 1) = 0$

$D = -1;\ D^2 = 1$

$\therefore \ D = \pm 1$

$\therefore$ D = –1, –1, –1

$\therefore$ $y_c = C_1 e^x + (c_2 + c_2 x) . e^{-x}$

Step 2: PI (y_P)

$$y_P = \frac{1}{D^3 + D^2 - D - 1}\cos 2x = \frac{1}{(D+1)(D^2-1)}\cos 2x$$

$$y_P = \frac{1}{(D+1)(-4-1)}\cos 2x \rightarrow D^2 = -4$$

$$= \frac{1}{(D+1)(-5)}\cos 2x = \frac{-1}{5}\frac{(D-1)}{(D+1)(D-1)}\cos 2x$$

$$= \frac{-1}{5}\frac{(D-1)}{D^2-1^2}\cos 2x = \frac{-1}{5}\frac{(D-1)}{(-4-1)}\cos 2x$$

$$= \frac{-1}{5}\frac{(D-1)\cos 2x}{-5} = \frac{1}{25}(D-1)\cos 2x$$

$$= \frac{1}{25}[D\cos 2x - \cos 2x] = \frac{1}{25}[-2\sin 2x - \cos 2x]$$

$$= \frac{-1}{25}(2\sin 2x + \cos 2x)$$

Step 3: General solution $= y_c + y_P$

$$= c_1e^x + (c_2 + c_3 x)\, e^{-x} + \frac{-1}{25}(2\sin 2x + \cos 2x)$$

Q.2(d) Solve $p^2 - py + x = 0$ **[5]**

Ans.: $p^2 - py + x = 0$

$\therefore$ $x = py - p^2$

Differentiate above w.r.t. y

$$\therefore \frac{1}{p} = p + y\frac{dp}{dy} - 2p\frac{dp}{dy} \qquad \therefore \frac{1-p^2}{p} = \frac{dp}{dy}(y-2p)$$

$$\therefore \frac{dy}{dx} - \frac{p}{1-p^2}y = \frac{-2p^2}{1-p^2}$$

It is a linear equation with "y" as dependent variable.

$$\text{I.F} = e^{-\int\frac{p}{1-p^2}dp} = e^{\frac{1}{2}\log\left(1-p^2\right)} = e^{\log\left(1-p^2\right)^{\frac{1}{2}}} = (1-p^2)^{\frac{1}{2}}$$

$$\therefore \text{I.F} = \sqrt{1-p^2}$$

$$\therefore y\sqrt{1-p^2} = -\int\frac{2p^2}{\sqrt{1-p^2}}dp + c \qquad \text{Put } p = \sin\theta$$

$$\therefore \int\frac{2p^2}{\sqrt{1-p^2}}dp = \sin^{-1} p\sqrt{1-p^2}$$

$$\therefore y = \frac{\sin^{-1}p}{\sqrt{1-p^2}} + p + \frac{c}{\sqrt{1-p^2}}$$

Thus x & y given by ;

$$x = py - p^2 \;\&\; y = \frac{\sin^{-1}p}{\sqrt{1-p^2}} + p + \frac{c}{\sqrt{1-p^2}}$$

Q.2(e) Solve $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 4y = \sin(\log x^2)$ **[5]**

Ans.: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 4y = \sin(\log x^2)$

$$\therefore x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 4y = \sin(2\log x)$$

Above equation is Cauchy's D.E.

Put $x = e^z$ $\therefore$ $z = \log x$ & $D = \frac{d}{dz}$

$\therefore$ D.E changes as :

$[D(D-1) + D + 4]\, y = \sin(2z)$

$\therefore$ $(D^2 + 4)\, y = \sin(2z)$

$\therefore$ A.E is $D^2 + 4 = 0$

$D^2 = 4$

$\therefore$ $D = \pm 2i$

$\therefore$ $y_c = c_1 \cos 2z + c_2 \sin 2z$

Now, $y_P = \frac{1}{D^2+4}\sin(2z) = \frac{z}{2D}\sin(2z) = \frac{z}{2}\int \sin(2z).dz$

$= \frac{z}{2}\frac{(-\cos 2z)}{2} = \frac{-z}{4}\cos(2z)$

$\therefore$ complete solution $= y_c + y_p$

$= c_1 \cos(2z) + c_2 \sin(2z) + \frac{-z}{4}\cos(2z)$

Q.2(f) Solve the Differential Equation $(x - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$ [5]

Ans.: $(x - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$

Now,

$M = x - 4xy - 2y^2$	$M = x - 4xy - 2y^2$
$\frac{\partial M}{\partial y} = -4x - 4y$	$\frac{\partial M}{\partial y} = -4x - 4y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ Given D.E is exact

Hence Solution $= \int M dx - \int (\text{Terms independent of } x \text{ is } N)\, dyzc$

$\therefore$ $\int (x - 4xy - 2y^2)\,dx + \int y^2 dy = c$

$\frac{x^2}{2} - 2x^2y - 2xy^2 + \frac{y^3}{3} = c$

Q.3Attempt the following (any THREE) [15]

Q.3(a) Evaluate $\int_0^\infty e^{-2t}\sin^2 t\,dt$ [5]

Ans.: $\sin^2 t = \frac{1-\cos 2t}{2}$

$\therefore$ $\frac{1}{2}\int_0^\infty e^{-2t}(1-\cos 2t)\,dt$ $\because$ $\frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2 4}\right]$

$\therefore$ $s = 2$

$\therefore$ $\frac{1}{2}\left[\frac{1}{2} - \frac{2}{4+4}\right]$

$\frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$

Q.3(b) Find inverse Laplace Transformation by convolution theorem for **[5]**

$$f(s) = \frac{s}{(s^2+1)(s^2+4)}$$

Ans.: Now, $\frac{1}{(s^2+1)} \times \frac{s}{s^2+4}$

cos 2t sin (t–u)

$$C.T. = \int_0^t \cos 2t \sin(t-u)\,du \qquad = \frac{1}{2}\int_0^t 2\cos 2t \sin(t-u)\,du$$

$$= \frac{1}{2}\left[\int_0^t \left[\sin(3t-u) - \sin(t+u)\right]du\right] \qquad = \frac{1}{2}\left[\frac{-\cos(3t-u)}{-1} + \frac{\cos(t+u)}{+1}\right]_0^t$$

$$= \frac{1}{2}\left[(\cos 2t - \cos 2t) - (\cos 3t - \cos t)\right] \quad = \frac{1}{2}(\cos t - \cos 3t)$$

Q.3(c) Find L[y(t) of the following differential equation: **[5]**

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = te^{-1};\ y(0) = 1 \text{ and } y'(0) = 2$$

Ans.:

$$\frac{d^2y}{dt^2} + \frac{2dy}{dt} + y = t\,e^{-t} \qquad \ldots (1)$$

Given : y(0) = 1 & y' (0) = 2

Taking L.T. on both the side in equation (1)

$$L\left[\frac{d^2y}{dt^2} + \frac{2\,dy}{dt} + y\right] = L\,[t\,e^{-t}]$$

$$s^2y(s) - s\,y(0) - y'(0) + 2\,[s\,y(s) - y(0)] + y(s) = \frac{1}{(s+1)^2} \qquad \Rightarrow \quad L[t^2] = \frac{1}{s^2}$$

$$(s^2 + 2s + 1)\,y\,(s) - s\,y(0) - y'\,(0) - 2y\,(0) = \frac{1}{(s+1)^2}$$

$$(s + 1)^2\,y\,(s) - s - 2 - 2 = \frac{1}{(s+1)^2}$$

$$\therefore \quad (s + 1)^2\,y\,(s) = \frac{1}{(s+1)^2} + (s+4)$$

$$\therefore \quad L\,[y(t)] = y(s) = \frac{1}{(s+1)^4} + \frac{(s+4)}{(s+1)^2}$$

Q.3(d) Find the inverse Laplace transform of : $\dfrac{5s+3}{(s+1)(s^2+2s+5)}$ **[5]**

Ans.:

$$\frac{5s+3}{(s+1)(s^2+2s+5)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+5}$$

$5s + 3 = A\,(s^2 + 2s + 5) + (B_s + C)\,(s + 1)$

$5s + 3 = As^2 + 2\,As + 5A + Bs^2 + Bs + Cs + C$

$5s + 3 = (A + B)\,s^2 + (2A + B + C)\,s + (5A + C)$

$\therefore$ A + B = 0; 2A + B + C = 5; 5A + C = 3

$$A = \frac{-1}{2}\ ;\ B = \frac{1}{2}\ ;\ C = \frac{11}{2}$$

$$\frac{5s+3}{(s+1)(s^2+2s+5)} = \frac{\frac{-1}{2}}{s+1} + \frac{\frac{1}{2}s + \frac{11}{2}}{s^2+2s+5}$$

$$= \frac{-1}{2}\left(\frac{1}{s+1}\right) + \frac{1}{2}\,\frac{s+11}{(s+1)^2+4} \quad = \frac{-1}{2}\left(\frac{1}{s+1}\right) + \frac{1}{2}\left[\frac{(s+1)+10}{(s+1)^2+4}\right]$$

$$= \frac{-1}{2}\left(\frac{1}{s+1}\right)+\frac{1}{2}\left[\frac{s+1}{(s+1)^2+4}+\frac{10}{(s+1)^2+4}\right]$$

$$\therefore \quad L^{-1}\left[\frac{5s+3}{(s+1)(s^2+2s+5)}\right] = L^{-1}\left\{\frac{1}{2}\left(\frac{1}{s+1}\right)+\frac{1}{2}\left[\frac{s+1}{(s+1)^2+4}+\frac{10}{(s+1)^2+4}\right]\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{1}{2}\left(\frac{1}{5}\right)+\frac{1}{2}\left(\frac{s}{s^2 4}\right)+5\left(\frac{1}{s^2+4}\right)\right\}$$

$$= e^{-t}\left\{\frac{1}{2}(1)\frac{1}{2}\cos(2t)+5.\frac{1}{2}\sin 2t\right\}$$

$$\therefore \quad L^{-1}\left[\frac{5s+3}{(s+1)(s^2+2s+5)}\right] = \frac{1}{2}e^{-t}(1+\cos 2t+5\sin 2t)$$

Q.3(e) Find the Laplace transform of : f(t) = $\begin{cases} 1 & 0 < t < a \\ -1 & a < t < 2a \end{cases}$ **and f(t) = f(t+2a)** **[5]**

Ans.: f (t) = f (t + 2a)

Above function is periodic function with period T = 2a

$\therefore$ LT of periodic function is given as :

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-su}f(u)du$$

$$\therefore \quad L[f(t)] = \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-su}(1)du+\int_a^{2a} e^{-su}(1)du\right] = \frac{1}{1-e^{-2as}}\left\{\left[\frac{e^{-su}}{-s}\right]_0^a-\left[\frac{e^{-su}}{-s}\right]_a^{2a}\right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[\frac{e^{-as}}{-s}\frac{-1}{-s}\right]-\left[\frac{e^{-2as}}{-s}-\frac{e^{-as}}{-s}\right]\right\}$$

$$= \frac{1}{(-s)(1-e^{-2as})}[e^{-as}-1-e^{-2as}+e^{-as}] \qquad = \frac{-e^{-2as}+2e^{-as}-1}{(-s)(1-e^{-2as})}$$

$$= \frac{1-2e^{-as}+e^{-2as}}{s(1-e^{-2as})} \qquad = \frac{(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})} \qquad = \frac{1-e^{-as}}{s(1+e^{-as})}$$

$$\therefore \quad L[f(t)] = \frac{1-e^{-as}}{s(1+e^{-as})}$$

Q.3(d) Obtain the inverse Laplace transform of each of the given function $\mathbf{\frac{(s+1)}{s^3(s-3)^2}}$ **[5]**

Ans.: Let; $\frac{s+1}{s^3(s-3)^2} = \frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s-3}+\frac{E}{(s-3)^2}$

$(s + 1) = As^2(s - 3)^2 + Bs(s - 3)^2 + c(s - 3)^2 + Ds^3(s - 3) + Es^3$

Assuming values of S we get,

$\therefore A = \frac{1}{9}$; $B = \frac{5}{27}$; $C = \frac{1}{9}$; $D = \frac{-1}{9}$; $E = \frac{4}{27}$

$$\therefore \frac{1/9}{s}+\frac{5/27}{s^2}+\frac{1/9}{s^3}-\frac{1/9}{s^3}+\frac{4/27}{(s-3)^2}$$

$$\therefore \frac{1}{9}+\frac{5}{27}t+\frac{1}{18}t^2-\frac{1}{9}e^{3t}+\frac{4}{27}te^{3t}$$

Q.4 Attempt any THREE of the following: [15]

Q.4(a) Evaluate : $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$ **[5]**

Ans.: Let $I = \int_0^1\int_0^1 \frac{dx\ dy}{\sqrt{(1-x^2)(1-y^2)}} = \int_0^1 \frac{1}{\sqrt{1-x^2}}\ dx.\ \int_0^1 \frac{dy}{\sqrt{1-y^2}}$

$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\ dx[\sin^{-1} y]_0^1 = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \left[\frac{\pi}{2}-0\right] = \frac{\pi}{2}\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$= \frac{\pi}{2}\ [\sin^{-1} x]_0^1 = \frac{\pi}{2}\left[\frac{\pi}{2}-0\right] = \frac{\pi^2}{4}$

Q.4(b) Evaluate $\int_0^3 \int_0^{\sqrt{1+y^2}} \frac{dx\,dy}{(1+x^2+y^2)}$ **[5]**

Ans.: $I = \int_0^3 \int_0^{\sqrt{1+y^2}} \frac{dx\,dy}{(1+y^2)+x^2}$

$= \int_0^3\int_0^A \frac{dx\,dy}{(A^2+x^2)}$ putting $\sqrt{1+y^2}$ = A for convenience

$= \int_0^3 \left[\frac{1}{A}\tan^{-1}\frac{x}{A}\right]_0^A dy = \int_0^3 \frac{1}{A}\left[\tan^{-1}\frac{A}{A} - \tan^{-1} 0\right]_0^A dy$

$= \int_0^3 \frac{1}{A}\left[\tan^{-1} 1 - 0\right] dy = \int_0^3 \frac{1}{A}\frac{\pi}{4} dy$

Now, $I = \frac{\pi}{4}\int_0^3 \frac{1}{\sqrt{1+y^2}}\ dy = \frac{\pi}{4}\left[\log[y+\sqrt{1+y^2}\,]\right]_0^3$

$= \frac{\pi}{4}\left[\log(3+\sqrt{10}) - \log 0\right] = \frac{\pi}{4}\log(3+\sqrt{10})$

Q.4(c) Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dx\,dy\,dz$ **[5]**

Ans.: $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dx\,dy\,dz$

Int. w.r.t. z

$= \int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y}\left[e^{z}\right]_0^{x+\log y}\ dx\,dy$

$= \int_0^{\log 2}\int_0^{x} e^{x+y}\left(e^{x+\log y} - e^0\right)dx\,dy = \int_0^{\log 2}\int_0^{x}\left(e^{2x}\,ye^{y} - e^{x}e^{y}\right)dx\,dy$

Int. w.r.t. y

$= \int_0^{\log 2} e^{2x}\left[ye^{y} - e^{y}\right]_0^{x} dx - \int_0^{\log 2} e^{x}\left[e^{y}\right]_0^{x} dx$

$= \int_0^{\log 2} e^{2x}\left[\left(xe^{x} - e^{x}\right) - \left(0 - e^{0}\right)\right] - \int_0^{\log 2} e^{x}\left[e^{x} - e^{0}\right] dx$

$= \int_0^{\log 2}\left(xe^{3x} - e^{3x} + 1\right)dx - \int_0^{\log 2}\left(e^{2x} - ex\right)dx$

$= \left[\frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + x - \frac{e^{3x}}{3} - \frac{e^{-2x}}{2} + e^{x}\right]_0^{\log 2}$

$= \log 2\frac{e^{3\log 2}}{3} - \frac{e^{3\log 2}}{9} + \log 2 - \frac{e^{3\log 2}}{3} - \frac{e^{2\log 2}}{2} + e^{\log 2}$

$$= \frac{8}{3}\log 2 - \frac{8}{9} + \log 2 - \frac{8}{3} - \frac{4}{2} + 2$$

$$= \log 2\left(\frac{8}{3} + 1\right) - \frac{8}{9} - \frac{8}{3}$$

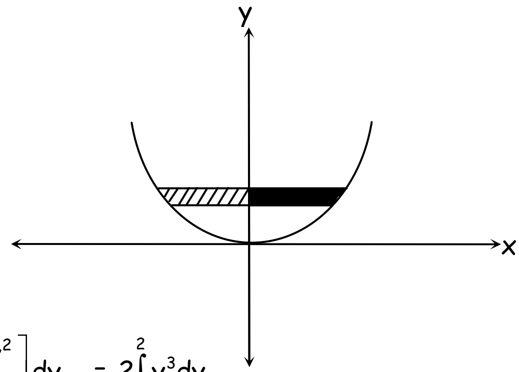
$$= \frac{11}{3}\log 2 - \frac{32}{3}$$

Q.4(d) Change the order of integration and evaluate $\int_0^2 \int_0^{x^2/4} xy\,dx\,dz$ **[5]**

Ans.: Here $y = \frac{x^2}{4}$ and y = 0

x = 0 to x = 2

$x^2 = 4y$ and y = 0



After change of order

$$= \int_0^2 \int_0^{2y} xy\,dx\,dy$$

Int. w.r.t. x

$$= \int_0^2 y\left[\frac{x^2}{2}\right]_0^{2y} dy \qquad = \int_0^2 y\left[\frac{4y^2}{2}\right] dy \qquad = 2\int_0^2 y^3 dy$$

$$= 2\frac{[y^4]_0^2}{4} \qquad = \frac{2\times 16}{4} \qquad = 8$$

Q.4(e) Evaluate $\iint y\,dx\,dy$ over the area bounded by $y = x^2$, $x + y = 2$ **[5]**

Ans.: Area bounded by $y = x^2$ (parabola) & $x + y = 2$

The point of intersection of $y = x^2$ & $x + y = 2$

$\therefore$ $x + x^2 = 2$

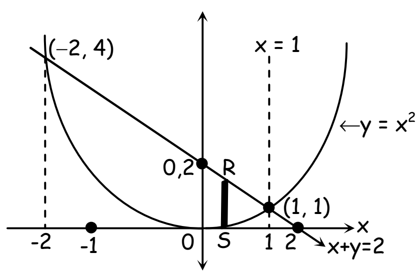
$\therefore$ $x^2 + x - 2 = 0$

$\therefore$ $x = 1, -2$

At x = 1 ; y = 1 & at x = −2; y = 4

$\therefore$ (1, 1) is the point of intersection in 1st quadrant. Take a vertical strip SR; along SR, x constant & y varies from to R i.e. $y = x^2$ to $y = 2 - x$

Now slide strip SR keeping 2nd quadrant to y − axis therefore y constant & x varies from x = 0 to x = 1



Step 2: $I = \int_0^1 \int_{x^2}^{2-x} y\,dx\,dy \qquad = \int_0^1 dx\left[\int_{x^2}^{2-x} y dy\right]$

$$= \int_0^1 dx . \frac{[y^2]_{x^2}^{2-x}}{2} \qquad = \frac{1}{2}\int_0^1 \left[(2-x)^2 - x^4\right] dx$$

$$= \frac{1}{2}\int_0^1 (4 - 4x + x^2 - x^4)\,dx$$

$$= \frac{1}{2}\left[4x - 2x^2 + \frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{2}\left[4 - 2 + \frac{1}{3} - \frac{1}{5}\right] \qquad = \frac{1}{2}\times\frac{32}{15}$$

$$I = \frac{16}{15}$$

Q.4(f) Evaluate $\int_0^3 \int_0^{\sqrt{1+y^2}} \frac{dx\,dy}{(1+x^2+y^2)}$ **[5]**

Ans.:

$$I = \int_0^3 \int_0^{\sqrt{1+y^2}} \frac{dx\,dy}{(1+y^2)+x^2}$$

$$= \int_0^3 \int_0^A \frac{dx\,dy}{(A^2+x^2)} \qquad \text{putting } \sqrt{1+y^2} = A \text{ for convenience}$$

$$= \int_0^3 \left[\frac{1}{A}\tan^{-1}\frac{x}{A}\right]_0^A dy \qquad = \int_0^3 \frac{1}{A}\left[\tan^{-1}\frac{A}{A} - \tan^{-1}0\right]_0^A dy$$

$$= \int_0^3 \frac{1}{A}\left[\tan^{-1}1 - 0\right]dy \qquad = \int_0^3 \frac{1}{A}\frac{\pi}{4}dy$$

Now, $$I = \frac{\pi}{4}\int_0^3 \frac{1}{\sqrt{1+y^2}}\,dy \qquad = \frac{\pi}{4}\left[\log[y+\sqrt{1+y^2}\,]\right]_0^3$$

$$= \frac{\pi}{4}\left[\log(3+\sqrt{10}) - \log 0\right] \qquad = \frac{\pi}{4}\log(3+\sqrt{10})$$

Q.5 Attempt the following (any THREE) **[15]**

Q.5(c) Evaluate $\int_0^{2a} x\left(2ax - x^2\right)^{1/2} dx$ **[5]**

Ans.: $\int_0^{2a} x\left(2ax - x^2\right)^{1/2} dx = \int_0^{2a} x^{3/2}\left(2a - x^2\right)^{1/2} dx$

Let $x = 2a\sin^2\theta$

$\therefore dx = 4a\sin\theta\cos\theta\,d\theta$

Where $x = 2a$; ………… $\theta \to \pi/2$

$x = 0$; $\theta \to 0$

$$= \int_0^{\pi/2} \left(2a\sin^2\theta\right)^{3/2}\left(2a - 2a\sin^2\theta\right)^{1/2} \times 4a\sin\theta\cos\theta\,d\theta$$

$$= 16a^3 \int_0^{\pi/2} \sin^3\theta\cos\theta\sin\theta\cos\theta\,d\theta \qquad = 16a^3\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta$$

$$= \frac{16a^3\,\Gamma\!\left(\frac{5}{2}\right)\Gamma\!\left(\frac{3}{2}\right)}{\Gamma 14} \qquad = \frac{16a^3 \times \frac{3}{2}\times\frac{1}{2}\Gamma\!\left(\frac{1}{2}\right)\times\frac{1}{2}\Gamma\!\left(\frac{1}{2}\right)}{2\times 16}$$

$$= \frac{16a^3\times 3\times\pi}{2\times 16} \qquad = \frac{16a^3\times 3\times\pi}{16\times 6} \qquad = \frac{\pi a^3}{2}$$

Q.5(b) Evaluate $\int_0^{\pi/2} \sin^6 x\cos^7 x\,dx$ **[5]**

Ans.: Where R = 6 and q = 7

$$\frac{\Gamma\!\left(\frac{R+1}{2}\right)\Gamma\!\left(\frac{q+1}{2}\right)}{2\,\Gamma\!\left(\frac{R+q+2}{2}\right)} = \frac{\Gamma\!\left(\frac{7}{2}\right)\Gamma 4}{2\,\Gamma(15/2)}$$

$$= \frac{\Gamma\!\left(\frac{7}{2}\right)\times 3}{2\times\frac{13}{2}\times\frac{11}{2}\times\frac{9}{2}\times\frac{7}{2}\Gamma\!\left(\frac{7}{2}\right)}$$

$$= \frac{8\times 6}{13\times 11\times 9\times 7} \qquad = \frac{16}{3003}$$

Q.5(c) Show that $\int_0^1 \frac{x^a - x^b}{\log x} \log\left(\frac{a+1}{b+1}\right)$ using DUIS. **[5]**

Ans.: "a" is a parameter & "b" is constant

Step 1: $I(a) = \int_0^1 \frac{x^a - x^b}{\log x} dx$ … (1)

Applying DUIS

$$I'(a) = \int_0^1 \frac{1}{\log x}\left[\frac{\partial}{\partial a}(x^a - x^b)\right]dx \quad \Rightarrow \left(\frac{\partial}{\partial a}x^b = 0 \text{ as } x^b \text{ is constant}\right)$$

$$I'(a) = \int_0^1 \frac{1}{\log x} x^a \log x \ dx \quad = \int_0^1 x^a \ dx \quad = \left[\frac{x^a + 1}{a+1}\right]_0^1 \quad = \frac{1}{a+1}$$

Step 2: $\therefore$ $I(a) = \int \frac{1}{a+1} da + A$

$\therefore$ $I(a) = \log(a + 1) + A$ …(2)

Step 3: Put a = b in equation (1)

$I(b) = 0$

Put a = b in equation (2)

$I(b) = \log(b + 1) + A$

$\therefore$ $A = -\log(b + 1)$

$\therefore$ From equation (2),

$I(a) = \log(a + 1) - \log(b + 1)$

$= \log\left(\frac{a+1}{b+1}\right)$ … Proved

Q.5(d) If $y = \int_0^x f(t)\sin[a(x - t)] \cdot dt$ then show that, $\frac{d^2y}{dx^2} + a^2y = af(x)$ **[5]**

Ans.: $\because$ x is a parameter

$\therefore$ $y = \int_0^x f(t)\sin[a(x-t)]dt$

Applying Leibnitz rule of DUIS w.r.t "x"

$$\frac{dy}{dx} = \int_0^x \frac{\partial}{\partial x}\{f(t)\sin[a(x-t)]\}dt + f(x)\sin a(x-x)\frac{d}{dx}x - f(0)\sin(ax)\frac{d}{dx}0$$

$$= \int_0^x f(t)a\cos[a(x-t)]dt + 0 - 0$$

$$\frac{dy}{dx} = a\int_0^x f(t)\cos[a(x-t)]dt$$

Again applying Leibnitz rule w.r.t. x

$$\frac{d^2y}{dx^2} = a\left[\int_0^x \frac{\partial}{\partial x} f(t)\cos[a(x-t)]dt + f(x)\cos[a(x-x)]\frac{d(x)}{dx} - 0\right]$$

$$= a\left[-\int_0^x f(t)\sin a(x-t)a\,dt + f(x)\right] \quad = -a^2\int_0^x f(t)\sin a(x-t)dt + a f(x)$$

$$\frac{d^2y}{dx^2} = -a^2y + a\,f(x) \qquad \left[\because y = \int_0^x f(t)\sin a(x-t)dt\right]$$

$\therefore$ $\frac{d^2y}{dx^2} + a^2y = a f(x)$ … Proved

Q.5(e) Evaluate : (i) erfc (–x) + erfc (x) (ii) erfc (x) + erfc (x) **[5]**

Ans.: (i) erfc (–x) + erfc (x)

w.k.t

erfc(x) + erfc(x)= 1

$\therefore$ erfc(x) = 1–erf(x)

$\therefore$ erfc(x) +1 – erf(x)

w.k.t.

erf(–x) = –erf(x)

$\therefore$ erfc(–x) + 1+erf(–x) = 2

(ii) erf(x) + erfc (x)

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\, dx \qquad \text{and} \qquad \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{x^2}\, dx$$

Now, $\text{erf}(x) + \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx + \frac{2}{\sqrt{\pi}} \int_x^\infty e^{x^2}$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{x^2} dx \quad = \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} \quad = 1$$

Q.5(e) Evaluate $\int_0^1 \frac{x^7}{(1-x^4)^{1/2}} dx$ **[5]**

Ans.: $\int_0^1 \frac{x^7}{(1-x^4)^{1/2}} dx$

Let $x^4 = t$; when x =1; t =1

$4x^3 dx = dt$ when x = 0; t = 0

$$= \frac{1}{4} \int_0^1 \frac{t\, dt}{(1-t)^{1/2}} \qquad = \frac{1}{4} \int_0^1 t^{2-1} (1-t)^{1/2-1}\, dt$$

$$= \frac{1}{4} \frac{\overline{|2}\,\overline{|1/2}}{\overline{|5/2}} \qquad = \frac{1}{4} \frac{\underline{|1} \times \overline{|1/2}}{3/2 \times 1/2 \times \overline{|1/2}}$$

$$= \frac{1}{3}$$

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