Numerical and Statistical Methods

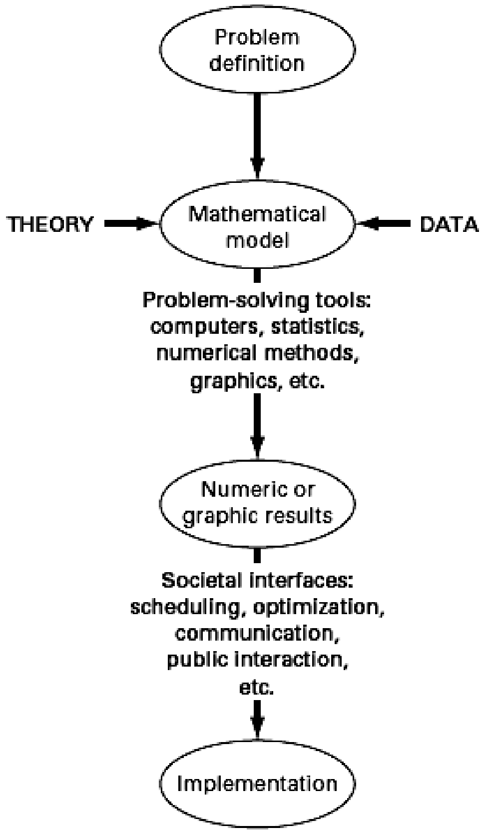
Time : $2\frac{1}{2}$ Hrs.] Prelim Question Paper Solution [Marks : 75

Q.1 Attempt any THREE of the following : [15]

Q.1(a) What is a mathematical model? With the help of a flowchart, explain the of solving an engineering problem. [5]

Ans.: A mathematical model can be broadly defined as a formulation or equation that expresses the essential features of a physical system or process in mathematical terms. In a very general sense, it can be represented as a functional relationship of the form :

$$\text{Dependent variable} = f\left(\begin{matrix}\text{indepdent}\\ \text{variables}\end{matrix}, \text{parameters}, \begin{matrix}\text{forcing}\\ \text{functions}\end{matrix}\right) \quad \ldots (1)$$



where the dependent variable is a characteristic that usually reflects the behavior or state of the system; the independent variables are usually dimensions, such as time and space, along which the system's behavior is being determined; the parameters are reflective of the system's properties or composition; and the forcing functions are external influences acting upon the system.

The actual mathematical expression of Eq. (1) can range from a simple algebraic relationship to large complicated sets of differential equations. For example, on the basis of his observations, Newton formulated his second law of motion, which states that the time rate of change of momentum of a body is equal to the resultant force acting on it. The mathematical expression, or model, of the second law is the well-known equation

$$F = ma \quad \ldots (2)$$

where F = net force acting on the body (N, or kg m/s^2),
m = mass of the object (kg), and
a = its acceleration (m/s^2)

Q.1(b) Explain the terms : (i) Significant figures, (ii) Accuracy, (iii) Precision, (iv) Truncation error, (v) Round-off error [5]

Ans.: **(i) Significant figures :** The concept of a significant figure or digit has been developed to formally designate the reliability of a numerical value. The significant digits of a number are those that can be used with confidence.

(ii) Accuracy : Accuracy refers to how closely a computed or measured value agrees with a true value.

(iii) Precision : Precision refers to how closely individual computed or measured values agree with each other.

(iv) Truncation error : These errors are generated when only required significant digits are retained in the number and remaining are discarded.

(v) Round-off error : Rounding off errors are errors arising from the process of rounding off during computation.

Q.1(c) Suppose that you have the task of measuring the lengths of a bridge and a rivet and come up with 9999 and 9 cm, respectively. If the true values are 10,000 and 10 cm, respectively, compute [5]
(i) the true error and
(ii) the true percent relative error for each case.

Ans.: (i) The error for measuring the bridge is

Error = |True value – Approximate value|

For bridge = |10,000 – 9,999| = 1 cm

For rivet = |10 – 9| = 1 cm

(ii) The percent relative error for the bridge is

$$(E_R) = \frac{(\text{True value} - \text{approx.value})}{\text{True value}}$$

$$(E_P) = \frac{(10{,}000 - 9{,}999)}{10{,}000} = \frac{1}{10{,}000} = 0.0001$$

% error $E_P = 0.0001 \times 100 = 0.01$ %

Relative Error for rivet

$$(E_R) = \frac{|10 - 9|}{10} = \frac{1}{10} = 0.1$$

% Error $(E_P) = 0.1 \times 100 = 10\%$

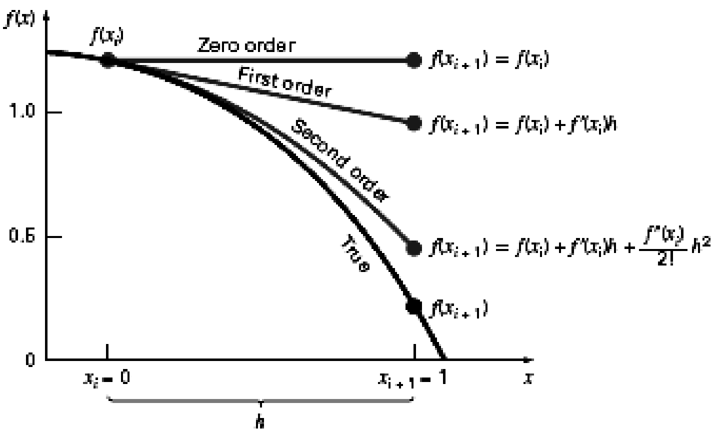
Q.1(d) Use zero-through fourth-order Taylor series expansions to approximate the function [5]

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

from $x_i = 0$ with $h = 1$. That is, predict the function's value at $x_{i+1} = 1$.

Ans.: Because we are dealing with a known function, we can compute values for f(x) between 0 and 1. The results indicate that the function starts at f(0) = 1.2 and then curves downward to f(1) = 0.2. Thus, the true value that we are trying to predict is 0.2.

The Taylor series approximation with n = 0 is $f(x_{i11}) = 1.2$



Q.1(e) Evaluate and interpret the condition number for **[5]**

$$f(x) = \frac{\sin x}{1+\cos x} \quad \text{for } x = 1.0001\pi$$

Ans.: $f(x) = \frac{\sin x}{1+\cos x} = \frac{u'v - vu'}{v^2}$ $u = \sin x$, $v = 1 + \cos x$

$x = 1.0001\,\pi$

$f(1.0001\,\pi) = -6366.2$

$f'(1.0001\,\pi) = 2.025 \times 10^7$

$\frac{f'(x)}{f(x)} = -10000.97$

Q.1(f) Evaluate e^{-5} using the two formulae : **[5]**

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$$

and $$e^{-x} = \frac{1}{e^x} = \frac{1}{1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots}$$

and compare with the true value 6.737947×10^{-3}. Use five terms to evaluate each series and compute true and approximate relative errors as terms are added.

Ans.: $e^{-5} = 1 - 5 + \frac{(5)^2}{2!} - \frac{5^3}{3!} + \frac{5^4}{4\times3\times2} = 1 - 5 + \frac{25}{2} - \frac{125}{6} + \frac{25\times25}{24} = 13.7083$

$e^{-5} = \frac{1}{e^5} = \frac{1}{1+5+\frac{5^2}{2!}+\frac{5^3}{3!}+\frac{5^4}{4!}} = \frac{1}{1+5+\frac{25}{2}+\frac{125}{6}+\frac{625}{4!}}$

$= \frac{1}{65.375} = 0.015296$

Error for first formula $= |\text{True value} - \text{Approximate value}|$
$= |6.737947 \times 10^{-3} - 13.70833|$
$= 37.015921$

Error while using second formula :
$= |\text{True Value} - \text{Approximate Value}|$
$= |6.737947 \times 10^{-3} - 0.0152961|$
$= 0.0085$

Q.2 Attempt any THREE of the following: **[15]**

Q.2(a) Find the roots of the equation $x^3 - 12.2x^2 + 7.45x + 42 = 0$ using Regula-Falsi method correct up to 4 decimal places. **[5]**

Ans.: $f(11) = (11)^3 - 12.2(11)^2 + 7.45(11) + 42$
$= 1331 - 12.2(121) + 7.45(11) + 42$
$= 1331 - 1476.2 + 81.95 + 42$
$= -21.25$

$f(12) = (12)^3 - 12.2(12)^2 + 7.45(12) + 42$
$= 1728 - 1756.8 + 89.4 + 42$
$= 102.6$

$f(11) \times f(12) < 0$ $a = 11, b = 12$

$x_1 = \frac{af[b] - bf[a]}{f[b] - f[a]} = \frac{11(102.6) - 12(-21.25)}{102.6 - (-21.25)}$

$= \frac{1128.6 + 255}{123.85} = \frac{1383.6}{123.85} = 11.1715$

x_1 = 11.1715 $f(x_1)$ = f(11.171)

$= (11.171)^3 - 12.2(11.171)^2 + 7.45(11.171) + 42$

$= 1394.0 - 1522.4 + 83.22 + 42$

$= -3.18$

$f(11.171) \times f(12) < 0$ a = 11.171, b = 12

f(a) = −3.18, f(b) = 102.6

$$x_2 = \frac{af[b]-bf[a]}{f[b]-f[a]} = \frac{(11.171)(102.6)-(12)(-3.18)}{102.6-(-3.18)}$$

$$= \frac{1146.14+38.16}{102.6+3.18} = \frac{1184.3}{105.78} = 11.1958$$

$f(x_2) = f(11.1958) = (11.195)^3 - 12.2(11.195)^2 + 7.45(11.195) + 42$

$= 1403.0 - 1529.0 + 83.40 + 42$

$= -0.6$

$f(11.195) \times f(12) < 0$, a = 11.195, b = 12

f(a) = −0.6, f(b) = 102.6

$$x_3 = \frac{af[b]-bf[a]}{f[b]-f[a]} = \frac{(11.195)(102.6)-(12)(-0.6)}{(102.6)-(-0.6)}$$

$$x_3 = \frac{1148.60+7.2}{103.2} = 11.1996$$

∴ $x_1 = 11.1715$

$x_2 = 11.1958$

$x_3 = 11.1996$

∴ approximate root = 11.1889 or 11.1990

Q.2(b) Find the roots of the equation x tan x = 1 near 4 using Newton Raphson method correct up to 4 decimal places. **[5]**

Ans.: $f(x) = x \cdot \tan x - 1$

$f'(x) = \tan x + x.\sec^2 x$

Given $x_0 = 4$

$f(x_0) = f(4) = x \tan x - 1 = 4 \tan 4 - 1$

$= 3.6312$

$f'(x_0) = \tan x + x.\sec^2 x$

$= 10.5200$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 4 - \left(\frac{3.6312}{10.5200}\right)$$

$x_1 = 3.6548$

$f(x_1) = f(3.6548) = x \tan x - 1$

$= 1.0597$

$f'(x_1) = f'(3.6548) = \tan x + x.\sec^2 x$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 13.6548 - \frac{1.0597}{5.3792}$$

$x_2 = 3.4578$

$f(x_2) = 0.1313$, $f'(x_2) = 4.1551$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3.4578 - \frac{0.1313}{4.1551}$$

$x_3 = 3.4256$

∴ $x_1 = 3.6548$, $x_2 = 3.4578$, $x_3 = 3.4256$

∴ approx. root = 3.4256

Q.2(c) **The following table shows the number of students and range of marks. Find the number of students who have secured less than 45 marks.** **[5]**

Marks	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of Students	31	45	32	27	15

Ans.:

Marks less than x	No. of Students
40	31
50	73
60	124
70	159
80	190

$h = x_1 - x_0 = 50 - 40 = 10$

$x = x_0 + hk$

$\therefore \; k = \dfrac{45-40}{10} \Rightarrow 0.5$

By Newtons forward difference Interpolation formula.

$$Y(x) = y_0 + k\,\Delta y_0 + \frac{k(k-1)}{2!}\Delta^2 y_0 + \frac{k(k-1)(k-2)}{3!}\Delta^3 y_0 + \frac{k(k-1)(k-2)(k-3)}{4!}\Delta^4 y_0$$

$$Y(45) = 31 + 0.5 \times 42 + \frac{0.5(0.5-1)}{2\times 1}\times 9 + \frac{0.5(0.5-1)(0.5-2)}{5\times 2\times 1}\times(-25) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!}\times 37$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31	42			
50	73	51	9	–25	
60	124	35	–16	12	37
70	159	31	–4		
80	190				

$\therefore \; y(45) = 47.868$

Q.2(d) **Given :** **[5]**

x	1	2	3	4	5	6	7	8
f(x)	0.01	0.004	0.02	0.12	0.15	0.257	0.325	0.231

find f(7.5) using Newton's backward interpolation formula.

Ans.: Newton's backward interpolation formula

$$Y(x) = x_n + \frac{k\nabla y_n}{1!} + \frac{k(k+1)}{2!}\nabla^2 y_n + \frac{k(k+1)(k+2)}{3!}\nabla^3 y_n + \ldots\ldots\ldots$$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$	$\nabla^6 y$	$\nabla^7 y$
1	0.01	–0.006	0.022	0.062	–0.216	0.517	–1.081	2.087
2	0.004	0.016	0.084	–0.154	0.301	0.564	1.006	
3	0.02	0.1	–0.07	0.147	–0.263	0.442		
4	0.12	0.03	0.077	–0.116	0.175			
5	0.15	0.107	–0.039	0.059				
6	0.257	0.068	0.02					
7	0.325	–0.088						
8	0.237							

$$Y(7.5) = 8+\frac{(-0.5)(-0.088)}{1!}+\frac{(-0.5)(-0.5+1)(0.02)}{2!}$$
$$+\frac{(-0.5)(-0.5+1)(-0.05+2)(0.059)}{3!}$$
$$+\frac{(-0.5)(-0.5+1)(-0.05+2)(-0.5+3)(0.175)}{4!}$$
$$+\frac{(-0.5)(-0.5+1)(-0.05+2)(-0.5+3)(-0.5+4)(0.442)}{5!}$$
$$+\frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)(-0.5+4)(-0.5+5)(1.006)}{6!}$$
$$+\frac{(-0.5)(-0.5+1)(-0.05+2)(-0.5+3)(-0.5+4)(-0.5+5)(-0.5+6)(2.087)}{7!}$$
$= 0.298$

Q.2(e) Determine the real root of $f(x) = 4x^3 - 6x^2 + 7x - 2.3$ using bisection method correct upto 3 decimal places. [5]

Ans.: $f(x) = 4x^3 - 6x^2 + 7x - 2.3$

$f(0) = 4(0)^3 - 6(0)^2 + 7(0) - 2.3 = -2.3$

$f(1) = 4(1)^3 - 6(1)^2 + 7(1) - 2.3 = 2.7$

$f(0) \times f(1) < 0$ Root lies in (0, 1)

Iterations are

x_1	x_2	$x_i = \frac{a+b}{2}$	$f(x_i)$
0.0000	1.0000	0.5000	0.2000
0.0000	0.5000	0.2500	−0.8625
0.2500	0.5000	0.3750	−0.3078
0.3750	0.5000	0.4375	−0.0509
0.4375	0.5000	0.4687	0.0748
0.4375	0.4381	0.4531	0.0120
0.4375	0.4531	0.4453	−0.0194
0.4458	0.4531	0.4451	−0.0036

The solution after successful iteration is 0.4451.

Q.2(f) *Given log 2 = 0.3010, log 3 = 0.4771, log 5 = 0.6990 and log 7 = 0.8451. Using Lagrange's formula, find log 47.* [5]

Ans.:

	x_0	x_1	x_2	x_3
x	log 2	log 3	log 5	log 7
y	0.3010	0.4771	0.6990	0.8451

f(x) by using Lagrange's Interpolation is given by :

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\times f(x_0)$$
$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\times f(x_1)$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\times f(x_2)$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\times f(x_3)$$

$$= \frac{(3.8501-1.0986)(3.8501-1.6094)(3.8501-1.9459)}{(0.6931-1.0986)(0.6931-1.6094)(0.6931-9.9459)}\times 0.30$$

$$+ \frac{(3.8501-0.6931)(3.8501-1.6094)(3.8501-1.9450)}{(1.0986-0.6931)(1.0986-1.6094)(1.0986-1.9459)}\times 0.4771$$

$$+ \frac{(3.8501-0.6931)(3.8501-1.0986)(3.8501-1.9459)}{(1.6094-0.6931)(1.6094-1.0986)(1.6094-1.9459)}\times 0.6990$$

$$+ \frac{(3.8501-0.6931)(3.8501-1.0986)(3.8501-1.6094)}{(1.9459-0.6931)(1.9459-1.0986)(1.9459-1.6094)}\times 0.8451$$

$= 1.6720$

Q.3 Attempt any THREE of the following: [15]

Q.3(a) Solve the following simultaneous equations by Gauss -Jordan elimination method: [5]

$2x_1 + 6x_2 - x_3 = -14$, $5x_1 - x_2 + 2x_3 = 29$, $x_3 - 3x_1 - 4x_2 = 4$

Ans.: $2x_1 + 6x_2 - x_3 = -14$ ...(1)

$5x_1 - x_2 + 2x_3 = 29$ ...(2)

$x_3 - 3x_1 - 4x_2 = 4$ ...(3)

Operate (2) $- \frac{5}{2}$(1) $\Rightarrow -16x_2 + \frac{9}{2}x_3 = 64$

Operate (3) + $\frac{3}{2}$(1) $\Rightarrow 5x_2 - \frac{1}{2}x_3 = -17$

The new reduced system of equations are

$2x_1 + 6x_2 - x_3 = -14$ ...(4)

$-16x_2 + \frac{9}{2}x_3 = 64$ ...(5)

$5x_2 - \frac{1}{2}x_3 = -17$ ...(6)

Operate (4) + $\frac{6}{16}$(5) $\Rightarrow 2x_1 + \frac{11}{16}x_3 = 10$

Operate (6) + $\frac{5}{16}$(5) $\Rightarrow \frac{29}{32}x_3 = 3$

$\therefore$ the new reduced system of equations are

$2x_1 + \frac{11}{16}x_3 = 10$ ...(7)

$-16x_2 + \frac{9}{2}x_3 = 64$ ...(8)

$\frac{29}{32}x_3 = 3$ ...(9)

Operate (7) $- \frac{\left(\frac{11}{16}\right)}{\left(\frac{29}{32}\right)}$(9) $\Rightarrow$ (7) $- \left(\frac{22}{29}\right)$(9)

$2x_1 = \frac{224}{29}$, $x_1 = \frac{112}{29}$

Operate (8) $- \left(\frac{144}{29}\right)$(9)

$-16x_2 = \frac{1424}{29}$

$\therefore \quad x_2 = \frac{-89}{29}$

$\frac{29}{32}x_3 = 3 \quad \therefore \quad x_3 = \frac{96}{29}$

∴ Final solution is

$x_1 = \frac{112}{29} = 3.8620$ $x_2 = \frac{-89}{29} = -3.689$ $x_3 = \frac{96}{29} = 3.3103$

Q.3(b) Solve the following simultaneous equations by Gauss-Seidel method : [5]
$10x_1 + x_2 + x_3 = 12$, $2x_1 + 10x_2 + x_3 = 13$, $2x_1 + 2x_2 + 10x_3 = 14$

Ans.: $x_1 = \frac{12 - x_2 - x_3}{10}$ … (1)

$x_2 = \frac{13 - 2x_1 - x_3}{10}$ … (2)

$x_3 = \frac{14 - 2x_1 - 2x_2}{10}$ … (3)

Iteration 1 : $x_2 = 0, x_3 = 0$

$(x_1)^0 = \frac{12 - 0 - 0}{10}$

$(x_1)^0 = 1.2$

Put $x_1 = 1.2$, $x_2 = 1.06$, $x_3 = 0$

$(x_2)^0 = \frac{13 - 2(12) - 0}{10} = 1.06$

$(x_3)^0 = \frac{14 - 2(1.2) - 2(1.06)}{10}$

$(x_3)^0 = 0.948$

Iteration 2: $(x_1)^1 = 0.9992$ $(x_2)^1 = 1.0053$ $(x_3)^1 = 0.999$

Iteration 3 : $(x_1)^2 = 0.9995$ $(x_2)^2 = 1.0001$ $(x_3)^2 = 1.0000$

After 3 iterations

$x_1 = 1$ $x_2 = 1$ $x_3 = 1$

Q.3(c) Evaluate $\int_0^1 \frac{1 - e^{-x}}{x} dx$ using trapezoidal rule and Simpson's 3/8 rule. [5]

Ans.: Let n = 5

Hence $h = \frac{(2-1)}{5} = 0.2$

$x_0 = 1$	1.2	1.4	1.6	1.8	2.0
$y_0 = 0.6321$	$y_1 = 0.5823$	$y_2 = 0.5381$	$y_3 = 0.4988$	$y_4 = 0.4637$	$y_5 = 0.4323$

Trapezoidal Rule

$\int_{x_0}^{x_1} y\,dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \ldots + y_{n-1}) + y_n]$

$= 0.51674$

By Simpson's 3/8th rule = 0.5110725

Q.3(d) Evaluate $\int_0^\pi \frac{\sin^2 x}{5 + 4\cos x} dx$ using Simpson's 3/8th rule. [5]

Ans.: Let n = 6

$x_0 = 0, x_n = \pi, h = \frac{x_n - x_0}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6}$

Simpson's $\left(\frac{3}{8}\right)^{th}$ rule

$$y = \frac{3h}{8}[(y_0 + y_n) + 2(\text{terms of y which are multiple of 3}) + 3(\text{remaining terms of y})]$$

$a = 0, b = \pi, n = 6$

x	0	π/6	π/3	π/2	2π/3	5π/6	π
y	0	0.0295	0.1070	0.1999	0.25	0.1632	0

$$y = \frac{3}{8}\times\frac{\pi}{6}\ [(0 + 0) + 2(\text{terms of y which one multiple of 3}) + 3(\text{remaining terms of y})]$$

$y = 0.4021$

Q.3(e) Solve $\frac{dy}{dx}$ = log (x + y); y(1) = 2 for x = 1.2 and x = 1.4 using Euler's modified method, taking h = 0.2. **[5]**

Ans.: $f(x, y) = \log(x + y)$

$y_0(1) = 2,\quad x_0 = 1,\ y_0 = 2,\ h = 0.2$

$f(x_0\ y_0) = \log(x_0 + y_0) = \log(1 + 2) = \log 3 = 1.0986$

To find y at x = 1.2

$y(1.2) = y_0 + f(x_0\ y_0)h = 2 + \log 3 \times 0.2 = 2.1972$

$f(x_1\ y_1) = f(1.2, 2.2172) = \log(1.2 + 2.172) = 1.2295$

$y(1.4) = y_1 + f(x_1\ y_1) * h = 2.2197 + 1.2295 * 0.2 = 2.465$

$\therefore\ y(1.4) = 2.465$

Q.3(f) Solve $\frac{dy}{dx} = \frac{y - x}{y + x}$, where y(0) = 1, to find y(0.1) using Runge-Kutta method. **[5]**

Ans.: $f(x, y) = \frac{y-x}{y+x}$

$x_0 = 0\qquad y_0 = 1$

$h = 0.1$

$k_1 = h.f(x_0, y_0) = 0.1\ f(0, 1)$

$k_1 = 0.1$

$k_2 = hf\ (x_0 + \frac{h}{2}, y_0 + k_1)$

$k_2 = 0.1\ f(0.05, 0.2)$

$k_2 = 0.06$

$k = \frac{1}{2}(k_1 + k_2)$

$k = 0.08$

$y_1 = y_0 + k$

$y_1 = 1 + 0.08$

$y_1 = 1.08$ at $x_1 = 0.1$

Q.4 Attempt any THREE of the following: **[15]**

Q.4(a) Fit a straight line to the x and y values in the two rows: **[5]**

x	1	2	3	4	5	6	7
y	0.5	2.5	2.0	4.0	3.5	6.0	5.2

Ans.: M = 7 which is odd

Let three required set line for best fit be

$Y = a + bx \qquad \ldots(1)$

Normal equations are

$\Sigma y = ma + b\,\Sigma x$ ...(2)

$\Sigma xy = a\Sigma x + b\Sigma x^2$ ...(3)

Consider the table no.

x	y	xy	x^2
1	0.5	0.5	1
2	2.5	5.0	4
3	2.0	6.0	9
4	4.0	16.0	16
5	3.5	17.5	25
6	6.0	36.0	36
7	5.2	36.4	49
$\Sigma x = 28$	$\Sigma y = 23.7$	$\Sigma xy = 118$	$\Sigma x^2 = 140$

Substituting in equation (2) and (3) we get

$23.7 = 7a + 28\,b$

$118.4 = 28\,a + 140\,b$

Solving we get

$a = 0.0145,\ b = 0.8428$

Required straight line is

$y = 0.0145 + 0.8428\,x$

Q.4(b) Fit a second degree parabola for the following : [5]

x	–2.5	–2	–1.5	–0.5	0	0.5	1.5
y	14.32	14.83	15.27	15.47	16.26	16.79	17.23

Ans.: Here $\Sigma x = -4.5$, $\Sigma y = 110.17$, $\Sigma x^2 = 15.25$

$\Sigma x^3 = -23.62$, $\Sigma x^4 = 65.31$

$\Sigma xy = -61.86$ $\Sigma x^2 y = 230.01$

$m = 7$

The equation of second degree parabola is

$y = a + bx + cx^2$

Solve $\Sigma y = na + b\Sigma x + c\Sigma x^2$

$\Sigma xy = a\,\Sigma\,x + b\,\Sigma\,x^2 + c\Sigma x^3$

$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^2 + c\Sigma x^4$

Substitute the values

$110.17 = 79 - 4.5b + 15.25c$ ...(1)

$-61.86 = -4.5a + 15.25b - 23.625c$ ...(2)

$230.01 = 15.25a - 23.625\,b + 65.3125c$ ...(3)

Solving we get, $a = 16.202,\ b = 0.726,\ c = 0.0013$

$\therefore$ second degree parabola is,

$y = 16.2029 + 0.7268x + 0.0013\,x^2$

Q.4(c) Use multiple regression to fit the following data : [5]

x_1	x_2	y
0	0	5
2	1	10
2.5	2	9
1	3	0
4	6	3
7	2	27

Ans.:

x_1	x_2	y	x^2	z^2	y_x	z_x	y_z
0	0	5	0	25	0	0	0
2	1	10	4	100	2	20	10
2.5	2	9	6.25	81	5	22.5	18
1	3	0	1	0	3	0	0
4	6	3	16	9	24	12	18
7	2	27	49	729	14	189	54

$\therefore$ $\sum x = 16.5$, $\sum y = 14$, $\sum z = 54$

$\sum x^2 = 76.25$, $\sum z^2 = 944$, $\sum xy = 48$

$\sum xz = 243.50$ $\sum yz = 100$

$\therefore$ $y = a + bx + cz$ —— required regression.

$\sum y = ma + b\sum x + c\sum z$

$\sum y\,x = a\sum x + b\sum x^2 + c\sum z\,x$

$\sum y\,z = a\sum z + b\sum z\,x + c\sum z^2$

Substitute the values in eq. (1), (2), (3)

$14 = 6a + 16.5b + 54c$ ...(1)

$48 = 16.5\,a + 76.25\,b + 243.5\,c$ ...(2)

$100 = 54\,a + 243.5\,b + 944\,c$ ...(3)

by solving equation (1), (2), (3) simultaneously and removing the variables we get, a = 1.667, b = 1.333, c = 0.333

Q.4(d) Maximize 50x + 100y subject to $10x + 5y \le 2500$, $4x + 10y \le 2000$, $x + 1.5y \le 450$ and $x \ge 0$; $y \ge 0$. **[5]**

Ans.: Maximize Z = 50x + 100y

Subject to $10x + 5y \le 2500$ ...(1)

$4x + 10y \le 2000$ ...(2)

$x + 1.5y \le 450$ ...(3)

and $x \ge 0$, $y \ge 0$

convert the given constraints into equations

$10x + 5y = 2500$

$4x + 10y = 2000$

$x + 1.5y = 450$

Consider 10x + 5y = 2500 we get

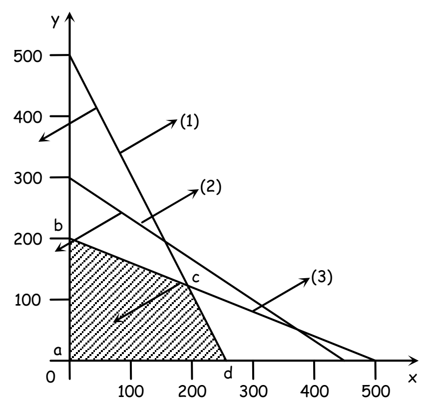
x	0	250
y	500	0

Consider 4x + 10y = 2000 we get

x	0	500
y	200	0

Consider 4x + 10y = 2000 we get

x	0	450
y	300	0



Feasible region is ABCDA

A(0, 0), B(0, 200),

C(200, 110) D(250, 0)

Points	Z = 50x + 100y
A	0
B	20000
C	21000
D	12500

Optimal Value of z is at C
X = 200 and Y = 110 is optimal solution.

Q.4(e) **A firm makes two types of furniture – chairs and tables. The contribution for each product as calculated by the accounting department is Rs. 20 per chair and Rs. 30 per table. Both products are processed on three machines M_1, M_2 and M_3 .The time required in hours by each product and total time available in hours per week on each machine are as follows:** **[5]**

MACHINE	CHAIR	TABLE	AVAILABLE TIME
M_1	3	3	36
M_2	5	2	50
M_3	2	6	60

How should the manufacturer schedule his production in order maximize contribution?

Ans.: Maximize Z = 20 x + 30y
Subject to
$3x + 3y \leq 36$
$5x + 2y \leq 50$
$2x + 6y \leq 60$
Non Negative constraints $x \geq 0, y \geq 0$

$3x + 3y \leq 36$

x	0	12
y	12	0

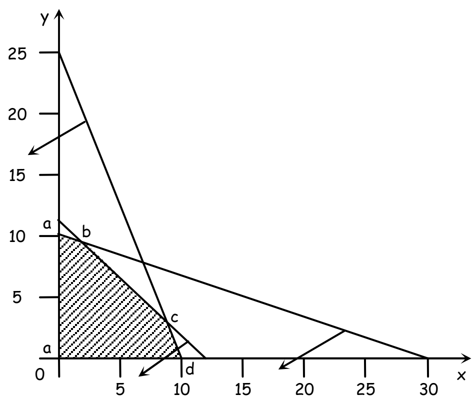
$5x + 2y \leq 50$

x	0	10
y	25	0

$2x + 6y \leq 60$

x	0	30
y	10	0

Vertex	Z = 20x + 30y
O = (0, 0)	0
A = (0, 10)	300
B = (5, 8)	340
C = (4, 5)	230
D = (10, 0)	200



The maximum contribution come upto point b = (5, 8) as x = 5 and y = 8

Q.4(f) **An aged person must receive 4000 units of vitamin, 50 units of minerals and 1400 calories a day. A dietician advises to thrive on two foods F1and F2 that cost Rs 4 and Rs 2 respectively per unit of food. It one unit of F1 contains 200 units of vitamins, 1 unit of mineral and 40 calories and one unit of F2 contains 100 units of vitamins 2 units of minerals and 40 calories, formulate a linear programming model to minimize the cost of diet.** **[5]**

Ans.:

Product	Food F1	Food F2	Requirements
Vitamins	200	100	4000
Minerals	1	2	50
Calories	40	40	40
Cost /Units	4	2	

$Z = 4x + 2y$

Subject to $200x + 100y \geq 4000$

$x + 2y \geq 50$

$40x + 40 \geq 1400$

$x, y \geq 0$

Q.5 Attempt any THREE of the following: [15]

Q.5(a) State and explain the properties of distribution functions. [5]

Ans.: Let x be a random variable, the function f defined for all real values x by $f(x) = P[X \leq x]$ for all real x is called distribution function. A distribution function is also called as cumulative probability distribution function.

Properties of distribution function :

1) If f is the distribution function of random variable x if a < b then
   $P[a < x \leq b] = f(b) - f(a)$
2) Values of all distribution function lies between 0 and 1 i.e.
   $0 \leq f(x) \leq 1$ for all x
3) All distribution functions are monotonically decreasing i.e. if x < y then f(x) < f(y).
4) $f(-\infty) = \lim_{x \to \infty} f(x) = 0$
   $f(-\infty) = \lim_{x \to \infty} f(x) = 1$
5) If x is discrete random variable then
   $f(x) = \sum_{x_i \leq x} P(x_1)$
6) If values of discrete random variable x are like $x_1 < x_2 < x_3 < x_4$ then
   $P(V_{n+1}) = f(x_{n+1}) - f(x_n)$
7) If x is discrete random variable then distribution function is step function.

Q.5(b) A random variable x has the following probability distributions : [5]

x	0	1	2	3	4	5	6	7
p(x)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2 + k$

(i) Find k, (ii) Evaluate P(x < 6), P (x ≥ 6) and P(0 < x < 5),

(iii) If P(x ≤ c) > $\frac{1}{2}$, find the minimum value of c, and

(iv) Determine the distribution function of x.

Ans.: i) Find k,

Sum of all probabilities is unity

$$\sum_{x=0}^{x=7} p(x) = 1$$

$$= 0 + k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$\therefore \quad k = \frac{1}{10}$$

ii) Evaluate P(x < 6)

$= P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3) + P(x= 4) + P(x = 5)$

$= 0 + k + 2k + 2k + 3k + k^2$

$= \frac{81}{100}$

iii) If $P(x \le c) > \frac{1}{2}$, find the minimum value of c

$P(x \le 0) = P(x = 0) = 0$ – False $\left(\text{not} > \frac{1}{2}\right)$

$P(x \le 1) = P(0) + P(x = 1) = 0 + \frac{1}{10} = \frac{1}{10} = 0.1 \left(\text{not} > \frac{1}{2}\right)$

$P(x \le 2) = P(0) + P(1) + P(2) = 0 + \frac{1}{10} + \frac{2}{10} = 0.3 \left(\text{not} > \frac{1}{2}\right)$

$P(x \le 3) = P(0) + P(1) + P(2) + P(3) = 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} = 0.5 \left(\text{not} > \frac{1}{2}\right)$

$P(x \le 4) = P(0) + P(1) + P(2) + P(3) + P(4)$

$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} = 0.8 \left(> \frac{1}{2}\right)$ condition is true.

Minimum value of c = 4

iv) $f(0) = P[x \le 0] = P(0) = 0$

$f(1) = P[x \le 1] = P(0) + P(1) = 0 + \frac{1}{10} = \frac{1}{10}$

$f(2) = P[x \le 2] = P(0) + P(1) + P(2)$

$= 0 + \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$

$f(3) = P[x \le 3] = P(0) + P(1) + P(2) + P(3)$

$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} = \frac{1}{2}$

$f(4) = P[x \le 4] = P(0) + P(1) + P(2) + P(3) + P(4)$

$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} = \frac{8}{10}$

$f(5) = P[x \le 5] = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$

$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{3}{10} + \frac{1}{100} + \frac{2}{100} = \frac{81}{100}$

$f(6) = \frac{83}{100}$; $f(7) = P(x \le 7) = 1$

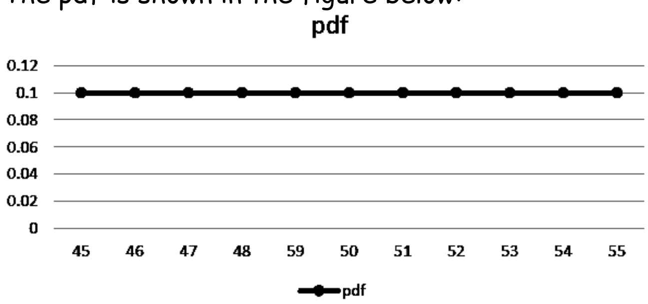
Q.5(c) The price for a litre of whole milk is uniformly distributed between Rs. 45 and Rs. 55 during July in Mumbai. Give the equation and graph the pdf for X, the price per litre of whole milk during July. Also determine the percent of stores that charge more than Rs. 54 per litre. **[5]**

Ans.: The equation of pdf is

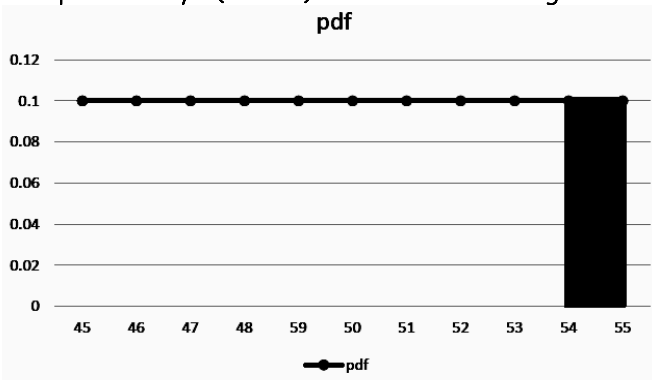
$f(x) = \frac{1}{55 - 45} = \frac{1}{10} = 0.1$ $\quad 45 < x < 55$

$= 0$ $\quad$ elsewhere

The pdf is shown in the figure below:



The probability P(X > 54) is shaded in the figure below:



The area of the shaded region is 0.1*1 = 0.1

Hence 0.10* 100 10% of the stores charge more than Rs. 54/– per litre.

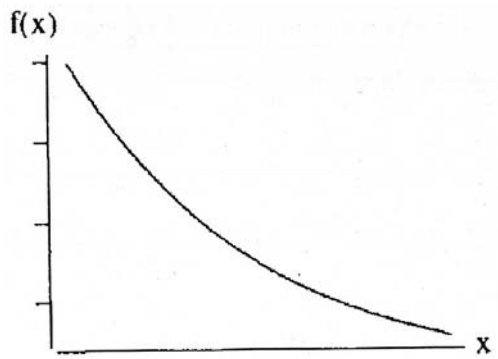
Q.5(d) What is exponential distribution? Suppose the time till death after infection with Cancer, is exponentially distributed with mean equal to 8 years. If X represents the time till death after infection with Cancer, then find the percentage of people who die within five years after infection with Cancer. **[5]**

Ans.: Exponential distribution :

The exponential probability distribution is a continuous probability distribution that is useful in describing the time it takes to complete some task. The pdf for an exponential probability distribution and e = 2.71828 five decimal places.

$$f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}} \quad \text{for } x \geq 0$$

The graph for the pdf of a typical exponential distribution is shown :



The percentage of people who die within five years after infection with cancer is

$P(X \leq 5) = 1 - e^{-\frac{5}{8}} = 1 - e^{-0.625} = 1 - 0.535 = 0.465$ i.e. 46.5%

Q.5(e) The probability mass function of a random variable X is zero except at the points i = 0, 1, 2. At these points it has the values p (0) = $3c^3$, p(1) = $4c - 10c^2$, p(2) = 5c − 1 for some c > 0. **[5]**

(i) Determine the value of c.

(ii) Compute the following probabilities, P (X < 2) and P (1 < X < 2).

(iii) Describe the distribution function and draw its graph.

(iv) Find the largest x such that F (x) < $\frac{1}{2}$

(v) Find the smallest x such that F (x) ≥ $\frac{1}{3}$

Ans.: Given : $P(0) = 3a^3$, $P(1) = 4a - 10a^2$ and $P(2) = 5a - 1$ and 0 for all the other values

(i) We know that if p(x) is a PMF, then $\sum_x P(x) = 1$

$\therefore \quad P(0) + P(1) + P(2) = 3a^3 + 4a - 10a^2 + 5a - 1 = 1$

$3a^3 - 10a^2 + 9a - 2 = 0$

$(a - 1)(3a^2 - 7a + 2) = 0$

$(a - 1)(3a - 1)(a - 2) = 0$

$\Rightarrow \quad a = 1, 2, \frac{1}{3}$

If a = 1, then P(0) = 3 > 1, which is not possible. Similarly, $1 \neq 2$

$\therefore \quad a = \frac{1}{3}$

$P(0) = 3\left(\frac{1}{3}\right)^3 = \frac{1}{9}$

$P(1) = 4a - 10a^2 = \frac{4}{3} - \frac{10}{9} - \frac{2}{9}$

$P(2) = 5a - 1 = \frac{5}{3} - 1 = \frac{2}{3} = \frac{6}{9}$

The probability distribution function is

x	0	1	2
P(X = x)	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{6}{9}$

**Consider a as c

(ii) $P(X < 2) = P(X = 0) + P(X = 1) = \frac{1}{9} + \frac{2}{9} = \frac{3}{9} = \frac{1}{3}$

$P(1 < X \leq 2) = P(X = 2) = \frac{6}{9} = \frac{2}{3}$

(iii) The distribution function is

$F(x) = 0, x < 0$

$= \frac{1}{9}, 0 < x < 1$

$= \frac{3}{9}, 1 \leq x < 2$

$= 1, x \geq 2$

(iv) Since $F(x) = \frac{1}{3} < \frac{1}{2}$, the largest value of x for which $f(x) < \frac{1}{2}$ is x = 1.

Since $F(x) = \frac{1}{3}$ for x = 1 and F(x) = 1 for $x \geq 2$, the smallest value of x for which $f(x) \geq \frac{1}{3}$ is x = 1.

Q.5(f) The monthly worldwide average number of airplane crashes of commercial airlines is 2.2. What is the probability that there will be **[5]**

(i) more than 2 such accidents in the next month?

(ii) more than 4 such accidents in the next 2 months?

Ans.: The number of crashes over a period of time is simply a random variable with a Poisson distribution. In this case, the sum of two passion random variables is just a new random variable with the new rate being the sum of the old rates.

(a) We have $X_1 \sim$ Poisson $(x_1 : \lambda_1 = 2.2)$; this is just

$P(X_1 > 2) = 1 - P(X_1 \leq 2)$

$= 1 - \left[e^{-22} + 2.2e^{-22} + \frac{(2.2)^2 e^{-2.2}}{2!}\right]$

$= 0.3772$

(b) Let X_2 be the number of accidents that happen in a two month period. By additively of the Poisson, $X_2 \sim$ Poisson $(x_2 : \lambda_2 = 2.2 + 2.2)$. Thus

$P(X_2 > 4) = 1 - P(X_2 \leq 4)$

$= 0.44881619145568419$

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