Numerical and Statistical Methods

Time : $2\frac{1}{2}$ Hrs.] Prelim Question Paper Solution **[Marks : 75**

Q.1 Attempt any THREE the following. [15]

Q.1(a) If true value of x = 1.732 and approximate value of x = 1.73 and $z = x_3 + x_2 - 1$. Then find the absolute, relative and percentage error in calculation of z. [5]

(A) Given : $x = 1.732$

$\bar{x} = 1.73$

$z = x^3 + x^2 - 1$

To find : $\left.\begin{array}{l} e_a = \\ e_r = \\ e_p = \end{array}\right\}$ in z

Solution : True value of z $= x^3 + x^2 - 1 = (1.732)^3 + (1.732)^2 - 1$

$= 7.19551917$

Approximate value of z $(\bar{z})$ $= x^3 + x^2 - 1 = (1.73)^3 + (1.73)^2 - 1$

$= 7.17061700$

(i) Absolute error (e_a) = |True value − Approximate value|

$= |z - \bar{z}|$

$= |7.19551917 - 7.17061700|$

$e_a = 0.02490217$

(ii) Relative error, $e_r = \dfrac{\text{Absolute Error}}{\text{True value}} = \dfrac{e_a}{z} = \dfrac{0.02490217}{7.19551917}$

$= 0.00346079$

(iii) Percentage error, $e_p = e_r \times 100\% = 0.00346079 \times 100\%$

$= 0.346079\%$

Q.1(b) Write the analytical solution of falling parachutist problem. [5]

(A) We shall use Newton's second law to determine the terminal velocity of a free falling body near the earths surface.

We know, $G = \dfrac{dV}{dt}$ … (1)

∴ eq. (1) becomes,

$$\frac{dV}{dt} = \frac{F}{m} \quad \ldots (2)$$

For a body falling within the vicinity of earth, the net force is composed of two opposing forces, the downward pull of gravity (F_D) and upward force of air resistance (F_U)

F →
- F_D (due to gravity) (assume +ve direction) ∴ $F_D = mg$ $g = 9.8 \text{ m/s}^2$
- F_U (due to air resistance) (assume −ve direction) ∴ $F_U \propto -V$, $F_U = -CV$

[∵ F is vector quantity, we need to assume +ve and −ve direction]

Upper force is linearly proportional to velocity, C ⇒ Drag's coefficient which takes into account shape and surface of object (jumpout) or orientation fo parachutist diving free fall.

∴ $F = F_D + F_U$

$F = mg - CV$ … (3)

Now, substituting eq. (2) in eq. (3)

$$\therefore \quad \frac{dV}{dt} = \frac{mg - CV}{m}$$

$$\frac{dV}{dt} = g - \frac{CV}{m}$$

Analytical solution :

$$V(t) = \frac{gm}{C}\left(1 - e^{-(C/M)t}\right) \qquad \ldots (4)$$

From the above equation (4)

V(t) – the dependent variable

t – independent variable

C & M – parameters

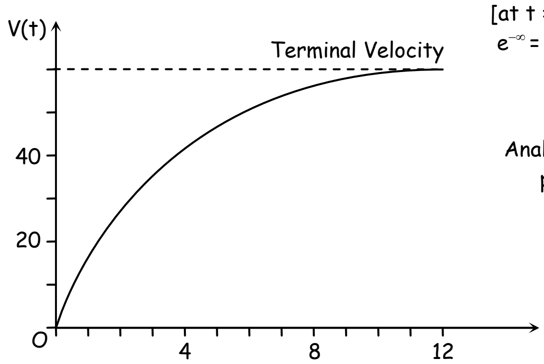
g – forming function

Now, consider a parachutist of mass 68.1 kg jumps out of a stationary hot air balloon use eq. (A) to compute velocity prior to opening the Chute. The drag coefficient is equal to 12.5 kg/s ($g = 9.8\ m/s^2$).

$$\therefore \quad V(t) = \frac{9.8(68.1)}{12.5}\left(1 - e^{-(12.5/65.81)t}\right)$$

$$V(t) = 53.39\ (1 - e^{-0.183557})$$

t(s)	V(t) (m/s)
0	0
2	16.4
4	27.77
6	35.64
8	41.10
10	44.87
12	47.49
:	:
∞	53.39



[at t = ∞ $e^{-\infty} = 0$]

Analytical solution to falling parachutest problem

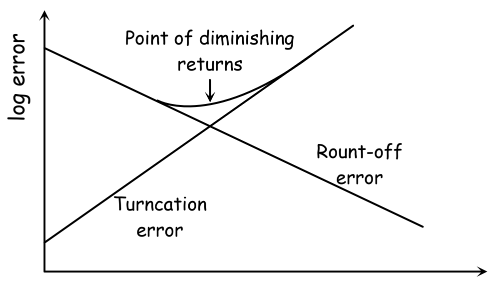
Thus velocity increases with time and asymptotically approaches a terminal velocity.

Eq. (A) is called analytical or exact solution because it exactly satisfies the origianl DE.

Q.1(c) Short note on : Total Numerical Errors. [5]

(A) Total Numerical Errors

It is summation of round-off and truncation error.



Trade-off between round-off and truncation error.

The point of diminishing returns is shown, where round-off error begins to negate the benefits of step size reduction.

From the above graph, we observe that the truncation error is minimum with smaller step size and increases linearly with the step size.

The round-off error is minimum with larger step size and hence, there si a trade-off between truncation error ε round-off error considering the step size.

We need to identify the point of diminishing returns which gives us the appropriate step size ε, where both the errors negate the effect of step size.

Q.1(d) Use Taylor series expansion where n = 0 to 3 to approximate f(x) = cos x at $x_{i+1} = \frac{\pi}{3}$ on the basis of the value of f(x) and its derivatives at $x_i = \frac{\pi}{4}$. **[5]**

(A) $h = x_{i+1} - x_i = \frac{\pi}{3} - \frac{\pi}{4}$

$h = \frac{\pi}{12}$

n = 0, zeroth order approximation

$f(x_{i+1}) \cong f(x_i) = f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = 0.7071$

n = 1, 1st order approximation

$f(x_{i+1}) \cong f(x_i) + h\,f'(x_i)$

n = 2, 2nd order approximation

$f(x_{i+1}) \cong f(x_i) + h\,f'(x_i) + \frac{h^2}{2!} f''(x_i)$

n = 3, 3rd order approximation

$f(x_{i+1}) \cong f(x_i) + h\,f'(x_i) + \frac{h^2}{2!} f''(x_i) + \frac{h^3}{3!} f'''(x_i)$

n	$f^n(x)$	$f(x_{i+1}) = f\left(\frac{\pi}{3}\right)$
0	cos x	0.7071
1	– sin x	0.5219
2	– cos x	0.4976
3	sin x	0.4997

Q.1(e) Find the round off error in storing the number 848.9735 using a four digit mantissa. **[5]**

(A) To find round-off error,

Given : x = 848.9735

d(length of mantissa) = 4

Chopping round-off

$x = 0.8489735 \times 10^3$

$x = f_x \times 10^E + g_x \times 10^{E-d}$

Approximate value, $\bar{x} = f_x \times 10^E$

Error $= g_x \times 10^{E-d}$

$\therefore \bar{x} = 0.8489 \times 10^3 = 848.9$

Error $= 0.735 \times 10^{3-4}$

$= 0.735 \times 10^{-1}$

Error = 0.0735

$\bar{x}$ = 848.9
Error = 0.0735

Symmetric Round-off

$g_x = 0.735 > 0.5$

Approximate value $(\bar{x}) = f_x \times 10^E + 10^{E-d} = 0.8489 \times 10^3 + 10^{3-4} = 848.9 + 0.1$

$\bar{x} = 849$

Error $= (g_x - 1) \times 10^{E-d} = (0.735 - 1) \times 10^{3-4} = -0.265 \times 10^{-1}$

Error = – 0.0265

$\therefore \bar{x} = 849$, Error = – 0.0265

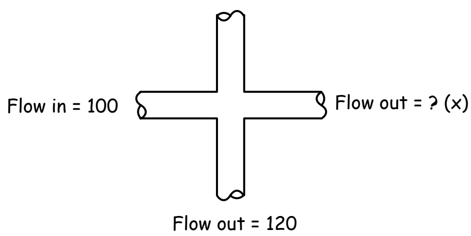
Q.1(f) Short note on Conservation laws and engineering problems. **[5]**

(A) Conservation Laws and Engineering Problems

Change = increases – decreases

1) Change varies with 't' ⇒ Transient state
2) No change, change = 0 ⇒ Steady state

Example : Fluid flow

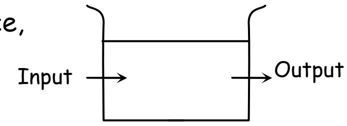
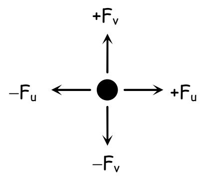
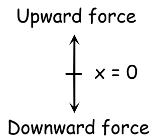
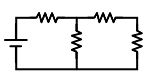
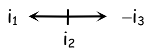


Flow in = Flow out

$100 + 80 = 120 + x$

$\therefore$ $x = 60$

Engineering Applications :

Field	Device	Organizing	Mathematical
1) Chemical Engineering	Reactors	Principle conservation of mass	Expression : Mass balance,  Δmass = input – output
2) Civil Engineering	Structure	Conservation of momentum	Force Balance,  At each node, $\sum F_u = 0$, $\sum F_v = 0$
3) Mechanical Engineering	Machine	Conservation of momentum	Force balance,  $m\frac{d^2x}{dt^2}$ = downward force – upward force
4) Electrical Engineering	Circuit 	Conservation of charge	Current balance,  For each mode, $\sum$current (i) = 0
		Conservation of energy	Votlage balance, $\sum$emfs – $\sum$votlage drop for resistors = 0

Q.2 Attempt any THREE the following. **[15]**

Q.2(a) Define the relation between E and Δ where E is shift operator and Δ is forward difference operator. Hence, find $\Delta^2(x^2)$ take h = 1. **[5]**

(A) Derive relation between E and Δ

And find $\Delta^2(x^2)$ take h = 1

Consider, Δ = forward difference operator

E = shift operator

We know $\Delta f(x) = f(x + h) - f(x)$ … (1)

and $E\,f(x) = f(x + h)$

$\therefore$ $(E - 1)(x) = E\,f(x) - f(x)$

$= f(x + h) - f(x)$ … (2)

From (1) and (2)

$\Delta f(x) = (E - 1) f(x)$

$\therefore$ $\Delta = E - 1$

Now $\Delta^2(x^2) = \Delta(\Delta x^2)$

$= \Delta((x + h)^2 - x^2)$ $(\because \Delta f(x) = f(x + h) - f(x))$

taking h = 1

$= \Delta((x + 1)^2 - x^2)$

$= \Delta(x^2 + 2x + 1 - x^2)$

$= \Delta(2x + 1)$

$= ((2(x + h) + 1) - (2x + 1))$

$\because$ h = 1 $= 2(x + 1) + 1) - 2x - 1)$

$= 2x + 2 + 1 - 2x - 1$

$\therefore \Delta^2(x^2) = 2$

Q.2(b) Find the missing term in the following table : **[5]**

x	0	1	2	3	4
y	1	3	9	–	8

(A)

x	0	1	2	3	4
y	1	3	9	–	8
	y_0	y_1	y_2	y_3	y_4

Here, h = 1

$\therefore \Delta^4 y_0 = 0$

We know, $\Delta = E - 1$

$(E - 1)^4 y_0 = 0$

$(E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 = 0$

$E^4 y_0 - 4E^2 y_0 + 6E^2 y_0 - 4E y_0 + y_0 = 0$

$\therefore y_{0+4(1)} - 4y_{0+3(1)} + 6y_{0+2(1)} - 4y_{0+1(1)} + y_0 = 0$

$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

$81 - 4y_3 + 6(3) - 4(3) + 1 = 0$

$4y_3 = 124$

$y_3 = 31$

$\therefore$ missing term, $y_3 = 31$

Q.2(c) Using bisection method find $\sqrt{30}$ approximately by performing 2 iterations. **[5]**

(A) Let $x = \sqrt[3]{20}$

$\therefore x^3 = 20$

$\therefore x^3 - 20 = 0$

$\therefore f(x) = x^3 - 20$

x = 0 $f(0) = -20 < 0$

x = 1 $f(1) = -19 < 0$

x = 2 $f(2) = -12 < 0$

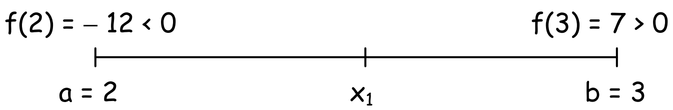
x = 3 $f(3) = 7 > 0$

$\therefore f(2) f(3) < 0$

$\therefore$ a = 2, b = 3

$\therefore$ Approximate root lies between a = 2 and b = 3

Bisection method, $x_n = \frac{a+b}{2}$



1^{st} iteration :

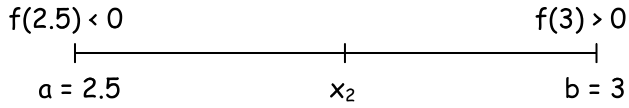
$x_1 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$

$\therefore f(x_1) = f(2.5) = (2.5)^3 - 20 = -4.375 < 0$

$\therefore$ $f(2.5)\, f(3) < 0$

$\therefore$ $a = 2.5 \qquad b = 3$

$\therefore$ Approximate root lies between $a = 2.5$, $b = 3$



2nd iteration :

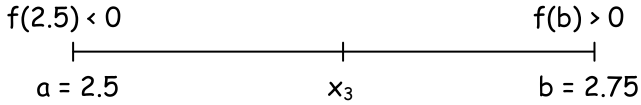
$$x_2 = \frac{a+b}{2} = \frac{2.5+3}{2} = 2.75$$

$$f(x_2) = f(2.75) = (2.75)^3 - 20 = 0.7968 > 0$$

$\therefore$ $f(2.5)\, f(2.75) < 0$

$\therefore$ $a = 2.5 \qquad b = 2.75$

Root lies between 2.5 and 2.75



3rd iteration :

$$x_3 = \frac{a+b}{2} = \frac{2.5+2.75}{2} = 2.625$$

$$f(x_3) = f(2.625) = (2.625)^3 - 20 = -1.912 < 0$$

$\therefore$ After 3rd iterations, the $\sqrt[3]{20} = 2.625$

Q.2(d) For the following data calculate f(0.25) using newton's interpolation formula. **[5]**

x	0.1	0.2	0.3	0.4	0.5
f(x)	1.4	1.56	1.76	2.00	2.28

(A) Given :

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 \Rightarrow 0.1$	$1.4 = y_0$			
		$0.16 = \Delta y_0$		
0.2	1.56		$0.04 = \Delta^2 y$	
		0.2		$0 = \Delta^3 y_0$
0.3	1.76		0.04	
		0.24		0
0.4	2.00		0.04	
		0.28		
0.5	2.28			

Using Newton forward Interpolation formula :

$$y = f(x) = y_0 + k\Delta y_0 + \frac{k(k-1)}{2!}\Delta^2 y_0 + \frac{k(K-1)(k-2)}{3!}\Delta^3 y_0 + \ldots$$

$\therefore$ $x = x_0 + Kh$

$x = 0.25, \quad x_0 = 0.1, \quad h = 0.1$

$$k = \frac{0.25-0.1}{0.1} = \frac{0.15}{0.1} = 1.5$$

$$\therefore\ y = f(0.25) = 1.4 + 1.5(0.16) + \frac{1.5(1.5-1)}{2}(0.04) = 0$$

$f(0.25) = 1.655$

Q.2(e) Given the following table. **[5]**

x	1	2	5	9
y = f(x)	1	3	6	10

Find f(6) using Lagrange's Interpolation formula.

(A)

	x_0	x_1	x_2	x_3
x	1	2	5	9
y = f(x)	1	3	6	10
	y_0	y_1	y_2	y_3

Find f(6)

Lagrange's Interpolation formula

$$y = f(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_3-x_2)}\times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\times y_3$$

Here x = 6

$$y = f(6) = \frac{(6-2)(6-5)(6-9)}{(1-2)(1-5)(1-9)}\times 1 + \frac{(6-1)(6-5)(6-9)}{(2-1)(2-5)(2-9)}\times 3$$

$$+ \frac{(6-1)(6-2)(6-9)}{(5-1)(5-2)(5-9)}\times 6 + \frac{(6-1)(6-2)(6-5)}{(9-1)(9-2)(9-5)}\times 10$$

$$f(6) = \frac{3}{8} + \left(\frac{-5}{7}\times 3\right) + \frac{15}{2} + \frac{25}{28}$$

$$= \frac{3}{8} - \frac{15}{7} + \frac{15}{2} + \frac{25}{28}$$

$$= \frac{3}{8} - \frac{15}{7} + \frac{15}{2} + \frac{25}{28}$$

f(6) = 6.625

Q.2(f) Solve the equation using Regula-Falsi method cos x − xe^x = 0 by performing two iterations. **[5]**

(A) Regula-Falsi method (2 iteration)

Given : $f(x) = \cos x - xe^x = 0$

$f(x) = \cos x - xe^x$

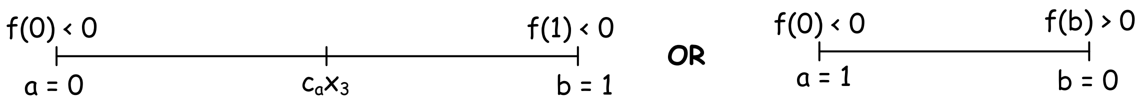
x = 0 $\quad f(0) = \cos 0 - 0 = 1 > 0 \qquad f(0) = 1$

x = 1 $\quad f(1) = \cos 1 - 1\,e^1 = -2.9779 < 0 \qquad f(0) = -2.1779$

$f(0)\,f(1) < 0$

∴ Approximate root lies between 0 and 1

a = 0, b = 1 *Or* a = 1 and b = 0



Using Regula-Falsi method

1st iteration.

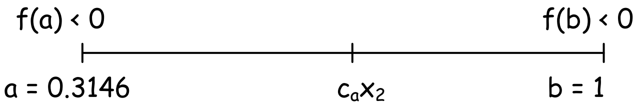
$$x_1 = C = \frac{af(b)-bf(a)}{f(b)-f(a)} = \frac{a\,f(1)-1\,f(0)}{f(1)-f(0)} = \frac{1}{2.1779-1} = 0.3146$$

$f(x_1) = f(c) = f(0.3146) = 0.52 > 0$

$f(0.3146) = 0.52$

$f(0.3146)\, f(1) < 0$

$\therefore$ Root lies between 0.3146 and 1 $\quad a = 0.3146, \quad b = 1$



2[st] iteration

$$x_2 = c_a = \frac{af(b)-bf(a)}{f(b)-f(a)} = \frac{0.3146\,f(1)-1\,f(0.3146)}{f(1)-f(0.3146)} = \frac{0.3146(-2.1779)-1(0.52)}{-21779-0.52}$$

$x_2 = C = 0.4467$

$f(x_2) = f(0.4467) = \cos(0.4467) = 0.4467$

$= 0.2036 > 0$

After 2 iteration the approximate root of f(x) is $x_2 = C = 0.4467$

Q.3 Attempt any THREE the following. [15]

Q.3(a) Use Gauss Jordan method to solve the following equation. [5]

$$2x_1 + 3x_2 - 4x_3 = 1$$
$$5x_1 + 9x_2 + 3x_3 = 17$$
$$-8x_1 - 2x_2 + x_3 = -9$$

(A) Use know AX = B

Gauss-Jordan methods

Reducing A $\rightarrow$ Digonal Matrix

AX = B

$$\begin{bmatrix} 2 & 3 & -4 \\ 5 & 9 & 3 \\ -8 & -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \\ -9 \end{bmatrix}$$

$R_2 : 2R_2 - 5R_1,\ \ R_3 = R_3 + 4R_1$

$$\begin{bmatrix} 2 & 3 & -4 \\ 0 & 3 & 26 \\ 0 & 10 & -15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 29 \\ -5 \end{bmatrix}$$

$R_3 : R_3/5$

$$\begin{bmatrix} 2 & 3 & -4 \\ 0 & 3 & 26 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 29 \\ -1 \end{bmatrix}$$

$R_1 : R_1 - R_3,\ \ R_3 : 3R_3 - 2R_2$

$$\begin{bmatrix} 2 & 0 & -30 \\ 0 & 3 & 26 \\ 0 & 0 & -61 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -28 \\ 29 \\ -61 \end{bmatrix}$$

$R_1/2, \quad R_3/-61$

$$\begin{bmatrix} 1 & 0 & -15 \\ 0 & 3 & 26 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -14 \\ 29 \\ 1 \end{bmatrix}$$

$R_1 : R_1 + 14R_3, \quad R_2 : R_2 - 26R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$\therefore \quad x_1 = 1$

$3x_2 = 3 \qquad x_2 = 1$

$x_3 = 1$

$x_1 = 1,\ x_2 = 1,\ x_3 = 1$

Q.3(b) Use Runge–kutta second elder formula to find y(0, 2). Taking h= 0.2 Given that $y(0) = 0$ and $\frac{dy}{dx} = 1 + y^2$. **[5]**

(A) Given $\frac{dy}{dx} = 1 + y^2$

$y(0) = 0$ i.e $x_0 = 0$

$y_0 = 0$

To find y(0.2) = ? using R.K. method of second order

Runge–kutta second order

$k_1 = h\, f(x_0\, y_0)$

$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + k_1\right)$

$\Delta_y = \frac{k_1 + k_2}{2}$

$y_{n+1} = y_n + \Delta y$

Here $x_1 = 0.2$, $y(0.2) = y(x_1) = y_1 = ?$

$h = x_1 - x_0 = 0.2 - 0 = 0.2$

$k_1 = 0.27(0.0) = 0.2(1) = 0.2$

$k_2 = 0.2\, f\left(0 + \frac{0.2}{2}, 0 + 0.2\right) = 0.2\, f(0.1, 0.2) = 0.2\,(1 + 0.3^2) = 0.208$

$\Delta y = \frac{k_1 + k_2}{2} = \frac{0.2 + 0.208}{2} = 0.204$

$\therefore\ y_1 = y_0 + \Delta y = 0 + 0.204$

$y(0.2) = 0.204$

Q.3(c) Use Trapezoidal rule to evaluate $\int_0^1 x^3 dx$. Considering fine sub-intervals. **[5]**

(A) (i) $y = f(x) = x^3$

(ii) $n = 5$

(iii) $a = 0$

(iv) $b = 1$

(v) $\therefore$ Step size $h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$

$h = 0.2$

(vi) Tabulate

x	0	0.2	0.4	0.6	0.8	1
y	0	0.08	0.064	0.216	0.512	1
	y_0	y_1	y_2	y_3	y_4	y_5

Using Trapezoidal Rule

$A = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \dots y_{n-1})\right]$

$\therefore\ A = \frac{0.2}{2}\left[(0+1) + 2(0.0008 + 0.064 + 0.216 + 0.512)\right]$

$= 0.1\,[1 + 1.6]$

$A = 0.26\ \text{unit}^2$

Q.3(d) Find the solution of the following system using Gauss Seidel Method. (perform two iteration only) **[5]**

$2x_1 + x_2 + x_3 = 5 \quad 3x_1 + 6x_2 + 2x_3 = 15 \quad 2x_1 + x_2 + 4x_3 = 8$

(A) Given : $2x_1 + x_2 + x_3 = 5$ … (1)

$3x_1 + 6x_2 + 2x_3 = 15$ … (2)

$2x_1 + x_2 + 4x_3 = 8$ … (3)

Gauss-Seidel method (2 iterations)

$\therefore$ From eq. (1) $x_1 = \frac{1}{2}(5 - x_2 - x_3)$

eq. (2) $x_2 = \frac{1}{6}(15 - 3x_1 - 2x_3)$

eq. (3) $x_3 = \frac{1}{4}(8 - 2x_1 - x_2)$

Initial conditions :

$x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$

1^{st} iteration :

$x_1^{(1)} = \frac{1}{2}\left(5 - x_2^{(1)} - x_3^{(1)}\right) = \frac{1}{2}(5 - 0.0) = \frac{5}{2} = 2.5$

$x_2^{(1)} = \frac{1}{6}\left(15 - 3x_1^{(1)} - 2x_3^{(0)}\right) = \frac{1}{6}\left(15 - 2(2.5) - 210\right) = 1.25$

$x_3^{(1)} = \frac{1}{4}\left(8.2x_1^{(1)} - x_2^{(1)}\right) = \frac{1}{4}\left(8 - 2(2.5) - 1.25\right) = 0.4375$

$x_1^{(1)} = 2.5, \quad x_2^{(1)} = 1.25, \quad x_3^{(1)} = 0.4375$

2^{nd} iteration :

$x_1^{(2)} = \frac{1}{2}\left(5 - x_2^{(1)} - x_3^{(1)}\right) = \frac{1}{2}(5 - 1.25 - 0.4375) = 1.65625$

$x_2^{(2)} = \frac{1}{6}\left(15 - 3x_1^{(2)} - 2x_3^{(1)}\right) = \frac{1}{6}\left(15 - 3(1.65625) - 2(0.4375)\right) = 1.526$

$x_3^{(2)} = \frac{1}{4}\left(8 - 2x_1^{(2)} - x_2^{(2)}\right) = \frac{1}{4}\left(8 - 2(1.65625) - 1.526\right) = 0.7903$

$\therefore$ After 2 iteration,

$x_1 = 1.65625, \quad x_2 = 1.526, \quad x_3 = 0.7903$

Q.3(e) Find $\frac{dy}{dx}$ at x = 6 given that **[5]**

x	4.5	5.0	5.5	6.0	6.5	7.0	7.5
y	9.69	12.9	16.71	21.18	26.37	32.34	39.15

(A) Using Newton Forward Interpolation,

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
	4.5	9.69				
			3.21			
	5.0	12.9		0.6		
			3.81		0.06	
	5.5	16.71		0.66		0
			4.47		0.06	
$x_0 \rightarrow$	6.0	$21.18 = y_0$		0.72		0
			$5.19 = \Delta y_0$		0.06	
	6.5	26.37		$0.78\ \Delta^2 y_0$		0
			5.97		$0.06\ \Delta^3 y_0$	
	7.0	32.34		0.84		
			6.81			
	7.5	39.15				

Here h = 0.5

$$\therefore \quad \left.\frac{dy}{dx}\right|_{x=0.6} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \ldots\right]$$

$$= \frac{1}{0.5}\left[5.19 - \frac{1}{2}(0.78) + \frac{1}{3}(0.06)\right]$$

$$\left.\frac{dy}{dx}\right|_{x=0.6} = 9.64$$

Q.3(f) Use Taylor series method, for the equation $\frac{dy}{dx} = x^2y$ and y(1) = 1 to find the value of y at x = 1.1, h=0.1 **[5]**

(A) Given $\frac{dy}{dx} = x^2y$

$y(1) = 1 \quad \therefore \quad x_0 = 1 \quad y_0 = 1$

$h = 0.1$

To find y(1.1) = ? $\quad x_1 = 1.1$ i.e. $y_1 = ?$

Solution :

Taylor Series

$$y(x_1) = y(x_0) + hy(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \ldots \quad \ldots (1)$$

$y' = \frac{dy}{dx}$ f(x, y) = x′y $\qquad y'(x_0) = x_0\, y_0 = 1$

$y''(x_n) = (1)(1) + 2(1)(1) = 3$

$y'' = x^2y' + 2xy$

$y''' = x^2y'' + 2xy' + 2xy' + 2y \qquad y'''(x_0) = (1)(3) + 4(1)(1) + 2(1) = 9$

$= x^2y'' + 4xy' + 2y$

$y^{iv} = x^2y''' + 2xy'' + 4xy'' + 4y + 2y'$

$= x^2y'' + 6xy'' + 6y' \qquad y^{iv}(x_0) = (1)(9) + 6(1)(3) + 6(1) = 33$

Substituting in equation (1)

$$\therefore \quad y(1.1) = 1 + 0.1(1) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(9) + \frac{(0.1)^4}{4!}(3) + \ldots\ldots$$

$y(1.1) = 1.11651$

Q.4 Attempt any THREE the following. **[15]**

Q.4(a) Calculate linear regression coefficient from the following data. **[5]**

x	1	2	3	4	5	6	7	8
y	3	7	10	12	14	17	20	24

(A) To obtain linear regression coefficients (b_{yx} and b_{xy})

x	y
1	3
2	7
3	10
4	12
5	14
6	17
7	20
8	24

Here n = 8

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} \qquad b_{xy} = r\frac{\sigma_x}{\sigma_y}$$

$$x = \frac{Cov(x, y)}{\sigma_x\sigma_y}$$

$$Cov(x, y) = E(xy) - E(x)\,E(y)$$

$$= \frac{\Sigma xy}{n} - \frac{\Sigma x}{n} - \frac{\Sigma y}{n}$$

$$\sigma_x = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$$

$$\sigma_y = \sqrt{\frac{\Sigma y^2}{n} - \left(\frac{\Sigma y}{n}\right)^2}$$

x	y	xy	x^2	y^2
1	3	3	1	9
2	7	14	4	49
3	10	30	9	100
4	12	48	16	144
5	14	70	25	196
6	17	102	36	289
7	20	140	49	400
8	24	192	64	576
$\sum x$ = 36	**$\sum y$ = 107**	**$\sum xy$ = 599**	**$\sum x^2$ = 204**	**$\sum y^2$ = 1763**

$$Cov = \frac{\Sigma xy}{n} - \frac{\Sigma x}{n}\frac{\Sigma y}{n} = \frac{599}{8} - \frac{36}{8}\times\frac{107}{8} = 74.875 - 60.1875$$

$= 14.6875$

$$\sigma_x = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2} = \sqrt{\frac{204}{8} - \left(\frac{36}{8}\right)^2} = \sqrt{5.25} = 2.2948$$

$$\sigma_y = \sqrt{\frac{\Sigma y^2}{n} - \left(\frac{\Sigma y}{n}\right)^2} = \sqrt{\frac{(1763)}{8} - \left(\frac{107}{8}\right)^2} = \sqrt{41.484} = 6.4408$$

$$\therefore\ r = \frac{Cov(x,y)}{\sigma_x \cdot \sigma_y} = \frac{14.6875}{2.29128 \times 6.4408} = 0.9952$$

$r = 0.9952$

Linear Regression coefficients

$$b_{yx} = r\cdot\frac{\sigma y}{\sigma x} = 0.9952 \times \frac{6.4408}{2.2912} = 2.7976$$

$$b_{xy} = \frac{r.\sigma_x}{\sigma_y} = 0.9952 \times \frac{2.29128}{6.4408} = 0.35402$$

$b_{yx} = 2.7976$, $b_{xy} = 0.3540$

Q.4(b) By the method of last sequences, fluid the straight line that best fits the following data. **[5]**

x	1	2	3	4	5
y	**14**	**27**	**40**	**55**	**68**

(A) Equation of straight line y = a + bx ...(A)

$\sum y = na + b\sum x$ ...(1)

$\sum xy = a\sum x + b\sum x^2$ ...(2)

Tabulate :

x	y	xy	x^2
1	14	14	1
2	27	54	4
3	40	120	9
4	55	220	16
5	68	340	25
$\sum x$ = 15	$\sum y$ = 204	$\sum xy$ = 748	$\sum x^2$ = 55

Substituting in equation (1) and equation (2)

Here n = 5

5a + 15 b = 204 ...(3)

15a + 55b = 748 ...(4)

solving equation (3) and equation (4) simultaneously.

$\therefore$ a = n b = 13.6

substituting the values of a & b in equation (A)

$\therefore$ Required straight line (y = 13.6 x)

Q.4(c) Diet for a sick person must contain atleast 4000 units of vitamin, 50 units of minerals and 1500 calories. Two foods F_1 and F_2 cost Rs. 50 and Rs. 75 per unit respectively. Each unit of food (F_1) contains 200 units of vitamins, 1 unit of minerals and 40 calories, whereas each unit of food F_2 contains 100 units of vitamins, 2 units of minerals and 30 calories. Formulate the L.P.P to satisfy sicker person's requirement at minimum cost. **[5]**

(A) Let x_1 be the number of units food f_1
and x_2 be the number of units food f_2

Objective function (z) minimize cost :
$z = 50 x_1 + 75x_2$

	Vitamins	Minerals	Calories
food f_1 (x_1)	200	1	40
food f_2 (x_2)	100	2	30
minimum Requirement	4000	50	1500

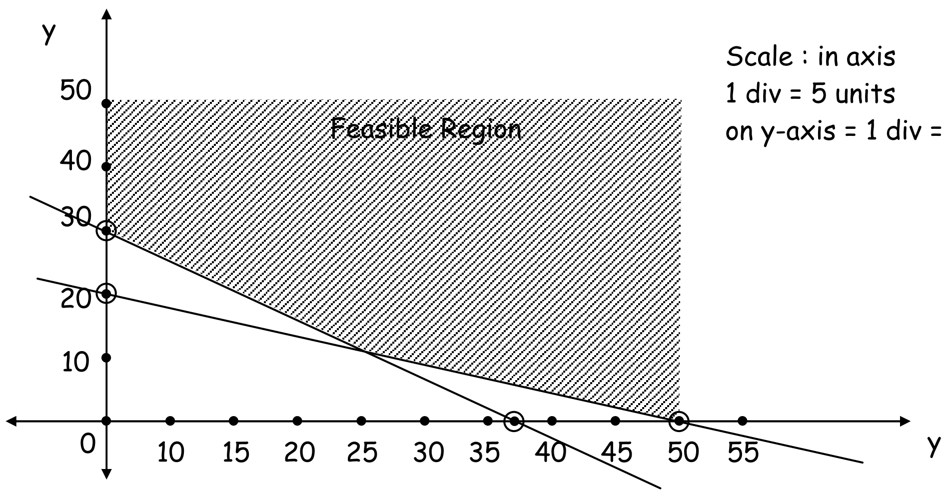
Formulation of L.P.P.
Minimise $Z = 50x_1 + 75x_2$
subject to $200x_1 + 100 x_2 \geq 4000$
$x_1 + 2x_2 \geq 50$
$40x_1 + 30x_2 \geq 1500$
Non-negative constructs $x_1 \geq 0$, $x_2 \geq 0$

Q.4(d) Solve graphically following LPP **[5]**
Minimise $z = 3x + 8y$
Subject to $3x + 10y \geq 150$
$4x + 5y \geq 150$
$x, y \geq 0$

(A) Minimise $z = 3x + 8y$
Subject to $3x + 10y \geq 150$
$4x + 5y \geq 150$
$x, y \geq 0$

Inequality constraints	**Equality constraints**	**Points**
$3x + 10y \geq 150$	$3x + 10y = 150$ ...(1)	A = (50, 0) B = (0, 15)
$4x + 5y \geq 150$	$4x + 5y = 150$ ...(2)	C = (37.5, 0) D = (0.30)

point of intersection of equation (1) and equation (2) x = 30, y = 6
solution using graphical method.



Scale : in axis
1 div = 5 units
on y-axis = 1 div = 10 units

point P (30.6)

Feasible region (DPA)

$z = 3x + 84$

D(0, 30) $z = 3(0) + 8(30) = 240$

P(30,6) $z = 3(30) + 8(6) = 138$ minimum

A(50,0) $z = 3(50) + 8(0)\ 150$

$\therefore$ z is minimum at point P(30,6) i.e. z = 138

$\therefore$ x = 30, y = 6

Q.4(e) The regression equation calculated from a set of observations for 2 variables are **[5]**

$x = -0.4y + 6.4$

$y = -0.6x + 4.6$

Find i) $\bar{x}$· ii) $\bar{y}$ iii) r

(A) Regression equations

$x = -0.4y + 6.4$ ...(1)

$y = -0.6x + 4.6$ ...(2)

To find : (1) r (2) $\bar{x}$ (3) $\bar{y}$

Regression of x on y

$(x-\bar{x}) = b_{xy}(y - \bar{y})$

$x = b_{xy}.y - \bar{y}b_{xy} + \bar{x}$

$\therefore x = b_{xy}.y + \bar{x} - \bar{y}b_{xy}$ ...(3)

Compare equation (1) & equation (3)

$\therefore b_{xy} = 0.4$

$\bar{x} - \bar{y}b_{xy} = 6.4$

$\bar{x} + 0.4\bar{y} = 6.4$ ...(A)

Regression of y on x

$(y-\bar{y}) = b_{xy}(x-\bar{x})$

$y = b_{yx}.x - \bar{x}b_{yx} + \bar{y}$ ...(4)

Compare equation (2) and equation (4)

$b_{yx} = -0.6$

$-\bar{x}b_{yx} + \bar{y} = 4.6$

$0.6\bar{x} + \bar{y} = 4.6$ ...(B)

Now correlation coefficient (r)

$r^2 = b_{xy} \times b_{yx} = -0.4 \times -0.6$

$r^2 = 0.24$

$\therefore r = \pm\sqrt{0.24} = -0.4899$ ($\because$ b_{xy} and b_{yx} are negative)

$r = -0.4899$

Solving equation (A) and equation (B) simultaneously.

$\therefore \bar{x} = 6$

$\bar{y} = 1$

Ans.: r = –0.4899 $\bar{x} = 6$ $\bar{y} = 1$

Q.4(f) Fit a least square quadratic curve to the following data estimate y(2 · 4) **[5]**

x	1	2	3	4
y	1.7	1.8	2.3	3.2

(A) Fit least square quadratic curve (parabolic) & estimate y(2.4)

Equation of parabola

$y = a + bx + cx^2$ ...(A)

$\sum y = na + b\sum x + c\sum x^2$ ...(1)

$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$ ...(2)

$\sum x^2 y = a\sum x^2 + b\sum x + c\sum x^4$ ...(3)

x	y	x^2	xy	x^3	x^2y	x^4
1	1.7	1	1.7	1	1.7	1
2	1.8	4	3.6	8	7.2	16
3	2.3	9	6.9	27	20.7	81
4	3.2	16	12.8	64	51.2	256
$\sum x = 10$	$\sum y=9$	$\sum x^2 = 30$	$\sum xy = 25$	$\sum x^3 = 100$	$\sum x^2y = 80.8$	$\sum x^4 = 354$

Here n = 4

$4a + 10\,b + 300 = 9$

$10a + 30b + 100c = 25$

$30a + 100b + 354c = 80.8$

$a = 2,\ b = \frac{-1}{2},\ c = \frac{1}{5}$

Substituting in equation (A)

$$y = 2+\left(\frac{-1}{2}\right)x+\left(\frac{1}{5}\right)x^2$$

Required equation of parabola.

$$y = 2-\frac{x}{2}+\frac{x^2}{5}$$

Q.5 Attempt any THREE the following. [15]

Q.5(a) If random variable x follows exponential distribution with parameter 0.5 find [5]

i) mean

ii) variance

iii) find 'a' such that P(x > a) = 0.4.

(A) x = exp. (2)

Given : parameter $(\lambda) = \frac{1}{2} = 0.5$

To find : i) mean ii) variance iii) find 'a' such that $P(x > a) = 0.4$

(i) $f(x) = \frac{1}{\lambda} = \frac{1}{\left(\frac{1}{2}\right)} = 2$

(ii) variance (var (x)) = $\frac{1}{\lambda^2} = \frac{1}{\left(\frac{1}{\lambda}\right)^2} = 4$

Exponential distribution

$f(x) = \lambda\, e^{-\lambda x}$ $x > 0$

$= 0$ $x < 0$

$$P(x > a) = \int_a^\infty fx(x)\,dx$$

$$0.4 = \int_a^\infty \lambda e^{-\lambda x}\,dx = \int_a^\infty \frac{1}{2}e^{-\frac{2}{2}}\,ax = \frac{1}{2}\left[\frac{e^{-\frac{2}{2}}}{\frac{1}{2}}\right]_a^\infty$$

$$0.4 = 2^{1\times 2}[e^{-a/2} - 0]$$

$\therefore$ $0.4 = e^{-a/1}$

Taking natural logarithm on both sides

$$\ell n(0.4) = -\frac{a}{2}\,\ell n(e)^0$$

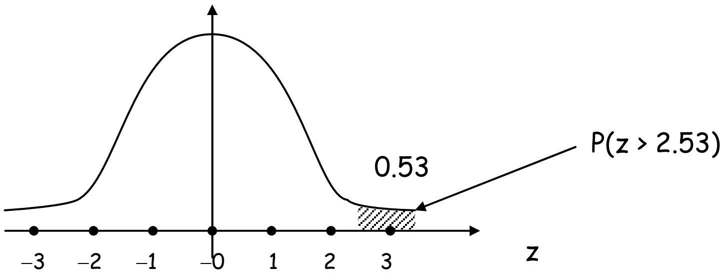
$$-0.9163 = -\frac{a}{2} \qquad a = 1.8326$$

Q.5(b) IF z ~ N (0, 1) find **[5]**

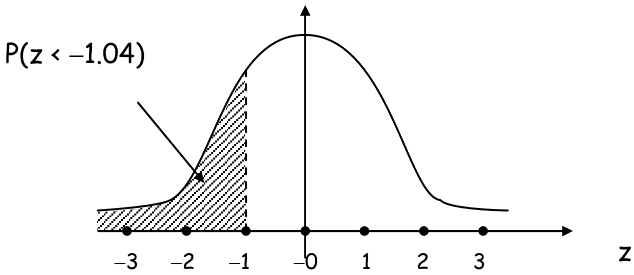
i) P(z > 2.53) **ii) P (z < –1.04)**

iii) P(–1.38 < z < 1.21) **iv) P (–2.12 < z < –2.08)**

(A)

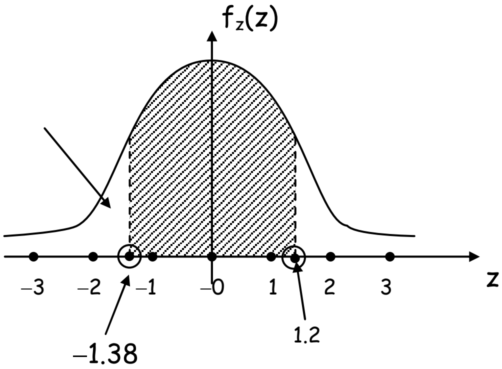


P(z > 2.53) = 0.5 – P(0 < z < 2.53)
= 0.5 – 0.4943
P(z > 2.53) = 0.0057

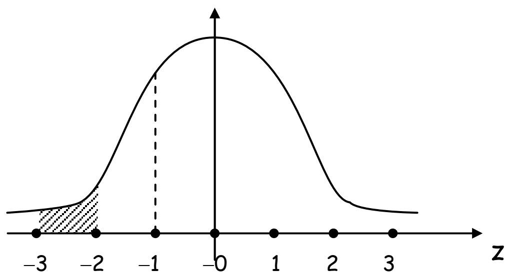


P(z < -1.04) = 0.5 – P(–1.04 < z < 0)
= 0.5 – 0.5308
P(z<–1.04) = 0.1492

(3)



(4)



P(1.38 < z < 1.21) = 0.4162 + 0.3869
P(–1.38 < z < 1.21) = 0.8031

P(–2.12 < z < –2.08)
= A(2.12) – A(2.08)
= 0.4930 – 0.4812 = 0.0118

∴ P(z > 2.53) = 0.0057
P(z < -1.04) = 0.1492
P(–1.38 < z < 1.21) = 0.8031
P(–2.12 < z < –2.08) = 0.0118

Q.5(c) The probability mass function of random variables x is given by.. **[5]**

$$P(X = x) = \frac{1}{8} \quad \text{If } x = 0$$
$$= \frac{1}{4} \quad \text{If } x = 1, 2, 3$$
$$= \frac{1}{8} \quad \text{If } x = 4$$

Find i) $P(x \le 1)$ ii) $P(x > 3)$ iii) $P(1 < x \le 3)$

(A) **Probability distribution**

x	0	1	2	3	4
P(x = x)	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$

(i) $P(x \le 1) = P(x = 0) + P(x = 1)$

$$= \frac{1}{8} + \frac{1}{4}$$

$$P(x \le 1) = \frac{3}{8}$$

(ii) $P(x > 3) = P(x = 4) = \frac{1}{8}$

(iii) $P(1 < x \le 3) = P(x = 2) + P(x = 3)$

$$= \frac{1}{4} + \frac{1}{4}$$

$$P(1 < x \le 3) = \frac{1}{2}$$

$\therefore$ $P(x \le 1) = \frac{3}{8}$ $\quad P(x > 3) = \frac{1}{8}$, $\quad P(1 < x \le 3) = \frac{1}{2}$

Q.5(d) Find the probability that at the most 5 defective bolts will be found in a box of 200 bolts, if it is known that 2 percent of such bolts are expected to be defective (given that $e^{-4} = 0.0183$) **[5]**

(A) Given : n = 200

P = probability of defective bulbs = 0.02

To find : $P(x \le 5)$

xV passion distribution (m)

$m = np = 200 \times 0.02 = 4$

$\therefore$ Poisson distribution is given by

$$P(x = x) = \frac{m^x . e^{-m}}{x!} \qquad x = 0.1...$$
$$= 0 \qquad \text{otherwise}$$

there m = 4

$\therefore$ $P(x = 5) = P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3) + P(x = 4) + P(x + 5)$

$$= \frac{4^0.e^{-4}}{0!} + \frac{4^1.e^{-4}}{1!} + \frac{4^2.e^{-4}}{2!} + \frac{4^3.e^{-4}}{3!} + \frac{4^4.e^{-4}}{4!} + \frac{4^5.e^{-4}}{5!}$$

$$= e^{-4}\left[4 + 4 + \frac{16}{2} + \frac{64}{6} + \frac{256}{24} + \frac{1024}{120}\right]$$

$P(x \le 5) = 0.784$

Q.5(e) fit a binomial distribution for the following data and hence find the expected frequencies. [5]

x	0	1	2	3	4
F	5	29	36	25	5

(A) (1) Here no of trails (n) = 4

Total frequencies N = $\sum f$ = 100

(2) Estimate ($\hat{p}$)

$$\bar{x} = \frac{\Sigma xf}{N} = \frac{196}{100} = 1.96$$

$$\hat{p} = \frac{\bar{x}}{n} = \frac{1.96}{4} = 0.49$$

$\therefore$ $\hat{q} = 1 - \hat{p} = 0.51$

$\therefore$ $P(X = x) = {}^nC_x\, p^n(\hat{q})^{n-x}$ $\quad x = 0, 1, \ldots 4$

$P(X = x) = {}^4C_x\,(0.99)^x\,(0.51)^{4-x}$ $\quad x = 0, 1, \ldots 4$

$= 0$ $\quad$ otherwise

$\therefore$ $P(X = x) = {}^4C_0\,(0.49)^0\,(0.51)^{4-0}$

$= (0.51)^4 = 0.06765$

3) Recurrence relation (RR)

$$P(X = x + 1) = \frac{\hat{p}}{\hat{q}}\left(\frac{n-x}{x+1}\right) P(X = x)$$

4) Tabulate :

x	$\frac{\hat{p}}{\hat{q}}\left(\frac{n-x}{x+1}\right)$	Use Recurrence Relation, $P(X = x + 1) = \frac{\hat{p}}{\hat{q}}\left(\frac{n-x}{x+1}\right) P(X = x)$	Expected frequencies E[F(x)] = NP
0	0.9603 (4) = 3.8431	0.06765	6.765 ≈ 7
1	$0.9609\left(\frac{3}{2}\right) = 1.441$	0.2579	25.79 ≈ 26
2	$0.9607\left(\frac{2}{3}\right) = 0.6404$	0.3745	37.45 ≈ 37
3	$0.9607\left(\frac{1}{4}\right) = 0.2401$	0.2398	23.98 ≈ 24
4	0	0.0575	5.75 ≈ 6
Total		0.9993 ≈ 1	N = 100

Q.5(f) If x is a normal variable using P(x < 60) = 0.4 and P(x > 80) = 0.2. Find mean and variance of x. [5]

(A) Given : $P(X < 60) = 0.4$

$X = N(\mu_1\sigma)$ $\quad P(X > 80) = 0.2$

Find : Mean (μ) = ?

Variance (σ^2) = ?

(i) $P(X < 60) = 0.4$

$$P\left(\frac{x-\mu}{\sigma} < \frac{65-\mu}{\sigma}\right) = 4$$

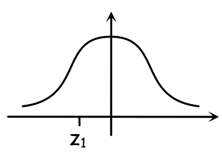
$P(z < z_1) = 0.4$ $\quad$ Let $z_1 = \frac{60-\mu}{\sigma}$ $\quad \ldots (1)$

$= 0.5 - 0.4$

$P(z < z_1) = 0.1$

$\therefore$ $z_1 = -0.25$ (from tables)

or $z_1 = -0.26$



(ii) $P(X > 80) = 52$

$P\left(\frac{x-\mu}{\sigma} > \frac{80-\mu}{\sigma}\right) = 0.2$

$P(z > z_2) = 0.2$ Let $z_1 = \frac{80-\mu}{\sigma}$ ... (2)

$= 0.5 - 0.2$

$P(z > z_2) = 0.3$

$z_2 = 0.84$
or $z_2 = 0.85$ } (from table)

From (1) $z_1 = \frac{60-\mu}{\sigma}$

$0.25 = \frac{60-\mu}{\sigma}$

$\mu - 0.265 = 60$ ... (3)

From (2) $z_2 = \frac{80-\mu}{\sigma}$

$0.64 = \frac{80-\mu}{\sigma}$

$\mu + 0.846 = 80$ ... (4)

Solve eq. (3) and eq. (4) simultaneously

$\mu = 64.5869$ $\sigma = 18348$ variance = σ^2

$\mu = 64.684$ $\sigma = 18.0.18$ Variance (σ) = 336.4

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