

Q.1 Attempt the following (any THREE)

[15]

Q.1(a) Verify Cayley-Hamilton theorem for the given matrix, also find inverse if exists.

[5]

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Ans.: Let;  $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$|A - \lambda I| = \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

Replace  $\lambda = A$ 

$$\therefore A^3 - 6A^2 + 9A - 4I = 0$$

$$A^2 = A \times A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \\ = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \\ + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

\(\therefore\) Cayley Hamilton Theorem is verified

Now;

$$A^3 - 6A^2 + 9A - 4I = 0$$

Pre Multiply above by  $A^{-1}$ 

$$\therefore A^{-1} \cdot A \cdot A^2 - 6A^{-1} \cdot A \cdot A + 9A^{-1} \cdot A - 4A^{-1} I = 0$$

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$\therefore 4A^{-1} = A^2 - 6A + 9I$$

$$\therefore A^{-1} = \frac{1}{4} [A^2 - 6A + 9I]$$

$$= \frac{1}{4} \left\{ \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ -5 & -5 & 6 \end{bmatrix} - 6 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$= \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Q.1(b) For different values of k, discuss the following equations:

[5]

$$x + 2y - z = 0; 3x + (k + 7)y - 3z = 0; 2x + 4y + (k - 3)z = 0$$

Ans.: Step 1:  $x + 2y - z = 0; 3x + (k + 7)y - 3z = 0; 2x + 4y + (k - 3)z = 0$

It can be written as;

$$Ax = z$$

$$\text{Where } A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & k+7 & -3 \\ 2 & 4 & k-3 \end{bmatrix}, x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, z = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Step 2: } [A|z] = \left[ \begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & k+7 & -3 & 0 \\ 3 & 4 & k-3 & 0 \end{array} \right]$$

$R_2 - 2R_1$  &  $R_3 - 3R_1$

$$= \left[ \begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & k+1 & 0 & 0 \\ 0 & 0 & k-1 & 0 \end{array} \right] \quad \dots(1)$$

Step 3: The system have infinite number of solutions if  $r = n$  i.e.  $P(A) < 3$  (number of unknown =  $n = 3$ ) From the last matrix it is possible only if  $k + 1 = 0$  or  $k - 1 = 0$  i.e.  $k = -1$  OR  $k = 1$

For  $k = 1$ :

$$[A|z] = \left[ \begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \dots \text{From (1)}$$

$$\therefore x + 2y - z = 0; 2y = 0$$

$$\therefore y = 0$$

$$\text{Put } z = k_1; -x = z = k_1 \quad \text{i.e. } x = -k_1; y = 0; z = k_1$$

For  $k = -1$ :

$$[A|z] = \left[ \begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array} \right] \quad \dots \text{From (1)}$$

$$x + 2y - z = 0; -2z = 0$$

$$\therefore z = 0; \quad x = -2y; \text{ Put } y = k_2$$

$$\text{Put } z = k_1 \quad \therefore x = -2k_2 \quad \text{i.e. } x = -2k_2; y = k_2 \text{ \& } z = 0$$

Q.1(c) Find the Characteristic values and characteristic vectors of the given matrix.

[5]

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Ans.: Characteristic Equation

$$A - \lambda I = \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$$

$$= (8 - \lambda)[(7 - \lambda)(3 - \lambda) - 16] + 6[-6(3 - \lambda) + 8] + 2[24 - 2(7 - \lambda)]$$

$$= -\lambda^3 + 18\lambda^2 - 45\lambda = 0$$

$$= \lambda^3 - 18\lambda^2 + 45\lambda = 0 \rightarrow (1)$$

Replace  $\lambda$  by "A" to get characteristics equation

$$\therefore A^3 - 18A^2 + 45A = 0$$

Roots of " $\lambda$ " from equation (1) is  $\lambda = 0, 3, 15$

(i) For  $\lambda = 0$

$$A - \lambda I = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 3x_3 = 0$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$\therefore x_1 = [1 \ 2 \ 2]^T$$

(ii) For  $\lambda = 3$

$$A - \lambda I = \begin{bmatrix} 5 & -6 & 2 \\ -6 & 2 & -4 \\ 2 & -4 & -2 \end{bmatrix}$$

$$5x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 2x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 - 2x_3 = 0$$

$$\therefore \frac{x_1}{-2} = \frac{x_2}{-1} = \frac{x_3}{2}$$

$$\therefore x_2 = [-2 \ -1 \ 2]^T$$

(iii) For  $\lambda = 15$

$$A - \lambda I = \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}$$

$$-7x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 - 8x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 - 12x_3 = 0$$

$$\therefore \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$\therefore x_3 = [2 \ -2 \ 1]^T$$

Q.1(d) Express  $\frac{-1}{2} + \frac{\sqrt{3}}{2}i$  in polar form.

[5]

Ans.: Consider,  $z = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$

Compare,  $z = x + iy$

$$\therefore x = \frac{-1}{2} \text{ \& } y = \frac{\sqrt{3}}{2}$$

Polar form of  $z = r [\cos \theta + i \sin \theta]$

... (1)

Where "r" is modulus &  $\theta$  is amplitude of z.

$$\text{Now, } |z| = r = \sqrt{x^2 + y^2}$$

$$\therefore r = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}}$$

$$\therefore r = 1$$

Amplitude of z :

$$x = \frac{-1}{2}, y = \frac{\sqrt{3}}{2} \quad x < 0 \text{ \& } y > 0$$

∴ It is in 2<sup>nd</sup> quadrant

We know ;

$$z = 0 = \pi - \tan^{-1} |y/x|$$

$$\therefore \theta = \pi - \tan^{-1} \left| \frac{\sqrt{3}/2}{-1/2} \right| = \pi - \tan^{-1}(\sqrt{3}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Substitute r & θ in equation (1)

$$\therefore z = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)$$

**Q.1(e) Prove that  $(1 + i\sqrt{3})^8 + (1 - i\sqrt{3})^8 = -2^8$**

[5]

**Ans.:** Let ;  $z_1 = 1 + i\sqrt{3}$  , = r [cos θ + i sin θ] ... (1)

Where; r is modules of z & θ is argument.

Modulus of z :

$$|z| = r \sqrt{x^2 + y^2} = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$$

Amplitude of z :

$$x > 0 \text{ \& } y > 0 \quad \therefore \text{ It is in 1<sup>st</sup> quadrant}$$

$$\therefore \theta = \tan^{-1} |y/x| = \tan^{-1} \pi/3$$

Substitute r & θ in equation (1)

$$\therefore z = 2 \left[ \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right]$$

by when  $z_2 = 1 - i\sqrt{3}$

$$\text{then } z_2 = 2 \left[ \cos \frac{\pi}{3} - i \sin \frac{\pi}{3} \right]$$

$$\begin{aligned} \text{Now, } z_1^8 + z_2^8 &= (1 + i\sqrt{3})^8 + (1 - i\sqrt{3})^8 \\ &= \left\{ 2 \left[ \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right] \right\}^8 + \left\{ 2 \left[ \cos \frac{\pi}{3} - i \sin \frac{\pi}{3} \right] \right\}^8 \end{aligned}$$

By De-Moivre's Theorem,

$$\begin{aligned} &= 2^8 \left[ \cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3} \right] + 2^8 \left[ \cos \frac{8\pi}{3} - i \sin \frac{8\pi}{3} \right] \\ &= 2^8 \left[ \cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3} + \cos \frac{8\pi}{3} - i \sin \frac{8\pi}{3} \right] \\ &= 2^8 \left[ 2 \cdot \cos \left( \frac{8\pi}{3} \right) \right] = 2^9 \cos \left( 3\pi - \frac{\pi}{3} \right) \\ &= 2^9 \left( -\cos \frac{\pi}{3} \right) = -2^9 \times \frac{1}{2} = -2^8 \end{aligned}$$

$$\therefore (1 + i\sqrt{3})^8 + (1 - i\sqrt{3})^8 = -2^8$$

**Q.1(f) Expand  $(1 + \cos x + i \sin x)^n$**

[5]

$$\begin{aligned} \text{Ans.} &= (2\cos^2 x/2 + 2\sin x/2 \cdot \cos x/2)^n \\ &= 2^n \cos^n x/2 (\cos x/2 - i \sin x/2)^n \\ &= 2^n \cos^n x/2 \left( \cos \frac{nx}{2} - i \sin \frac{nx}{2} \right) \rightarrow \text{By D.T.} \end{aligned}$$

Q.2 Attempt the following (any THREE)

[15]

Q.2(a) Solve the Differential Equation  $dy / dx + x^2y = x^5$

[5]

Ans.:  $\frac{dy}{dx} + x^2y = x^5$

Above equation is in Linear Equation form

i.e.  $\frac{dy}{dx} + Py = Q$        $P = x^2, \theta = x^5$

I.F =  $e^{\int P dx} = e^{\int x^2 dx} = e^{x^3/3}$

Solution:

$Y \times \text{I.F} = \int Q \times \text{I.F} dx$

$y \times e^{x^3/3} = \int x^5 \cdot e^{x^3/3} dx$

Let  $\frac{x^3}{3} = t \therefore x^3 = 3t \quad \therefore x^3 dx = dt$

$y \times e^{x^3/3} = \int 3t e^t dt = 3 [te^t - e^t] + c = 3 \left[ \frac{x^3}{3} e^{x^3/3} - e^{x^3/3} \right] + c$

$y \times e^{x^3/3} = [x^3 e^{x^3/3} - 3e^{x^3/3}] + c$

Q.2(b) Solve the following Equation  $x^2p^2 - 2xpy + (2y^2 - x^2) = 0$

[5]

Ans.:  $x^2p^2 - 2xpy + (2y^2 - x^2) = 0$

Solvable by "P"

Compare with  $ap^2 + bp + c$

$a = x^2; \theta = -2xy; c = 2y^2 - x^2$

$P = \frac{2xy \pm \sqrt{4x^2y^2 - 4x^2(2y^2 - x^2)}}{2x^2} = \frac{2xy \pm \sqrt{4x^4 - 4x^2y^2}}{2x^2}$

$P = \frac{y}{x} \pm \sqrt{1 - \frac{y^2}{x^2}}$

$\frac{dy}{dx} = \frac{y}{x} \pm \sqrt{1 - \frac{y^2}{x^2}}$

Let  $\frac{y}{x} = v \therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$

$v + x \frac{dv}{dx} = v \pm \sqrt{1 - v^2}$

$\frac{dv}{\sqrt{1 - v^2}} = \pm \frac{dx}{x}$

Integrating on both the side

$\int \frac{dv}{\sqrt{1 - v^2}} = \pm \int \frac{dx}{x}$

$\sin^{-1}v = \pm \log x + c$

$\sin^{-1}\left(\frac{y}{x}\right) = \pm \log x + c$

Q.2(c) Solve  $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \cos 2x$

[5]

Ans.: The given DE is  $(D^3 + D^2 - D - 1)y = \cos 2x$

Step 1: C.F ( $y_c$ )

$y_c = \text{A.E is } D^3 + D^2 - D - 1 = 0$

$\therefore D^2(D+1) - 1(D+1) = 0 \Rightarrow (D^2 - 1)(D+1) = 0$

$D = -1; D^2 = 1$

$\therefore D = \pm 1$

$$\begin{aligned} \therefore D &= -1, -1, -1 \\ \therefore y_c &= C_1 e^x + (c_2 + c_2 x) \cdot e^{-x} \end{aligned}$$

Step 2: PI ( $y_p$ )

$$\begin{aligned} y_p &= \frac{1}{D^3 + D^2 - D - 1} \cos 2x &= \frac{1}{(D+1)(D^2-1)} \cos 2x \\ y_p &= \frac{1}{(D+1)(-4-1)} \cos 2x \rightarrow D^2 = -4 \\ &= \frac{1}{(D+1)(-5)} \cos 2x &= \frac{-1}{5} \frac{(D-1)}{(D+1)(D-1)} \cos 2x \\ &= \frac{-1}{5} \frac{(D-1)}{D^2-1^2} \cos 2x &= \frac{-1}{5} \frac{(D-1)}{(-4-1)} \cos 2x \\ &= \frac{-1}{5} \frac{(D-1) \cos 2x}{-5} &= \frac{1}{25} (D-1) \cos 2x \\ &= \frac{1}{25} [D \cos 2x - \cos 2x] &= \frac{1}{25} [-2 \sin 2x - \cos 2x] \\ &= \frac{-1}{25} (2 \sin 2x + \cos 2x) \end{aligned}$$

Step 3: General solution =  $y_c + y_p$   
 $= c_1 e^x + (c_2 + c_3 x) e^{-x} + \frac{-1}{25} (2 \sin 2x + \cos 2x)$

**Q.2(d) Solve  $p^2 - py + x = 0$**

[5]

Ans.:  $p^2 - py + x = 0$

$$\therefore x = py - p^2$$

Differentiate above w.r.t.  $y$

$$\therefore \frac{1}{p} = p + y \frac{dp}{dy} - 2p \frac{dp}{dy} \quad \therefore \frac{1-p^2}{p} = \frac{dp}{dy} (y-2p)$$

$$\therefore \frac{dy}{dx} - \frac{p}{1-p^2} y = \frac{-2p^2}{1-p^2}$$

It is a linear equation with "y" as dependent variable.

$$\text{I.F} = e^{-\int \frac{p}{1-p^2} dp} = e^{\frac{1}{2} \log(1-p^2)} = e^{\log(1-p^2)^{\frac{1}{2}}} = (1-p^2)^{\frac{1}{2}}$$

$$\therefore \text{I.F} = \sqrt{1-p^2}$$

$$\therefore y \sqrt{1-p^2} = -\int \frac{2p^2}{\sqrt{1-p^2}} dp + c \quad \text{Put } p = \sin \theta$$

$$\therefore \int \frac{2p^2}{\sqrt{1-p^2}} dp = \sin^{-1} p \sqrt{1-p^2}$$

$$\therefore y = \frac{\sin^{-1} p}{\sqrt{1-p^2}} + p + \frac{c}{\sqrt{1-p^2}}$$

Thus x & y given by :

$$x = py - p^2 \quad \& \quad y = \frac{\sin^{-1} p}{\sqrt{1-p^2}} + p + \frac{c}{\sqrt{1-p^2}}$$

**Q.2(e) Solve  $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + 4y = \sin(\log x^2)$**

[5]

Ans.:  $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + 4y = \sin(\log x^2)$

$$\therefore x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + 4y = \sin(2 \log x)$$

Above equation is Cauchy's D.E.

$$\text{Put } x = e^z \quad \therefore z = \log x \text{ \& } D = \frac{d}{dz}$$

$\therefore$  D.E changes as :

$$[D(D - 1) + D + 4] y = \sin(2z)$$

$$\therefore (D^2 + 4) y = \sin(2z)$$

$$\therefore \text{A.E is } D^2 + 4 = 0$$

$$D^2 = -4$$

$$\therefore D = \pm 2i$$

$$\therefore y_c = c_1 \cos 2z + c_2 \sin 2z$$

$$\text{Now, } y_p = \frac{1}{D^2 + 4} \sin(2z) = \frac{z}{2D} \sin(2z) = \frac{z}{2} \int \sin(2z) \cdot dz$$

$$= \frac{z}{2} \frac{(-\cos 2z)}{2} = \frac{-z}{4} \cos(2z)$$

$$\therefore \text{complete solution} = y_c + y_p$$

$$= c_1 \cos(2z) + c_2 \sin(2z) + \frac{-z}{4} \cos(2z)$$

**Q.2(f) Solve the Differential Equation  $(x - 4xy - 2y^2) dx + (y^2 - 4xy - 2x^2) dy = 0$**

[5]

**Ans.:**  $(x - 4xy - 2y^2) dx + (y^2 - 4xy - 2x^2) dy = 0$

Now,

$$\begin{array}{l} M = x - 4xy - 2y^2 \\ \frac{\partial M}{\partial y} = -4x - 4y \end{array} \quad \left| \quad \begin{array}{l} M = x - 4xy - 2y^2 \\ \frac{\partial M}{\partial x} = -4x - 4y \end{array} \right.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$  Given D.E is exact

Hence Solution =  $\int M dx - \int (\text{Terms independent of } x \text{ is } N) dy + c$

$$\therefore \int (x - 4xy - 2y^2) dx + \int y^2 dy = c$$

$$\frac{x^2}{2} - 2x^2y - 2xy^2 + \frac{y^3}{3} = c$$

**Q.3 Attempt the following (any THREE)**

[15]

**Q.3(a) Evaluate  $\int_0^{\infty} e^{-2t} \sin^2 t \, dt$**

[5]

**Ans.:**  $\sin^2 t = \frac{1 - \cos 2t}{2}$

$$\therefore \frac{1}{2} \int_0^{\infty} e^{-2t} (1 - \cos 2t) dt \quad \therefore \frac{1}{2} \left[ \frac{1}{s} - \frac{s}{s^2 + 4} \right]$$

$$\therefore s = 2$$

$$\therefore \frac{1}{2} \left[ \frac{1}{2} - \frac{2}{4 + 4} \right]$$

$$\frac{1}{2} \left( \frac{1}{2} - \frac{1}{4} \right) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

Q.3(b) Find inverse Laplace Transformation by convolution theorem for

[5]

$$f(s) = \frac{s}{(s^2 + 1)(s^2 + 4)}$$

Ans.: Now,  $\frac{1}{(s^2 + 1)} \times \frac{s}{s^2 + 4}$

$$\cos 2t \sin(t-u)$$

$$C.T. = \int_0^t \cos 2t \sin(t-u) du = \frac{1}{2} \int_0^t 2 \cos 2t \sin(t-u) du$$

$$= \frac{1}{2} \left[ \int_0^t [\sin(3t-u) - \sin(t+u)] du \right] = \frac{1}{2} \left[ \frac{-\cos(3t-u)}{-1} + \frac{\cos(t+u)}{+1} \right]_0^t$$

$$= \frac{1}{2} [(\cos 2t - \cos 2t) - (\cos 3t - \cos t)] = \frac{1}{2} (\cos t - \cos 3t)$$

Q.3(c) Find L[y(t)] of the following differential equation:

[5]

$$\frac{d^2y}{dt^2} + 2 \frac{dy}{dt} + y = te^{-t}; y(0) = 1 \text{ and } y'(0) = 2$$

Ans.:  $\frac{d^2y}{dt^2} + \frac{2dy}{dt} + y = te^{-t} \dots (1)$

Given :  $y(0) = 1$  &  $y'(0) = 2$

Taking L.T. on both the side in equation (1)

$$L \left[ \frac{d^2y}{dt^2} + \frac{2dy}{dt} + y \right] = L [te^{-t}]$$

$$s^2y(s) - sy(0) - y'(0) + 2[sy(s) - y(0)] + y(s) = \frac{1}{(s+1)^2} \Rightarrow L[t^2] = \frac{1}{s^2}$$

$$(s^2 + 2s + 1)y(s) - sy(0) - y'(0) - 2y(0) = \frac{1}{(s+1)^2}$$

$$(s+1)^2 y(s) - s - 2 - 2 = \frac{1}{(s+1)^2}$$

$$\therefore (s+1)^2 y(s) = \frac{1}{(s+1)^2} + (s+4)$$

$$\therefore L[y(t)] = y(s) = \frac{1}{(s+1)^4} + \frac{(s+4)}{(s+1)^2}$$

Q.3(d) Find the inverse Laplace transform of :  $\frac{5s+3}{(s+1)(s^2+2s+5)}$

[5]

Ans.:  $\frac{5s+3}{(s+1)(s^2+2s+5)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+5}$

$$5s+3 = A(s^2+2s+5) + (Bs+C)(s+1)$$

$$5s+3 = As^2 + 2As + 5A + Bs^2 + Bs + Cs + C$$

$$5s+3 = (A+B)s^2 + (2A+B+C)s + (5A+C)$$

$$\therefore A+B=0; 2A+B+C=5; 5A+C=3$$

$$A = \frac{-1}{2}; B = \frac{1}{2}; C = \frac{11}{2}$$

$$\frac{5s+3}{(s+1)(s^2+2s+5)} = \frac{-1}{2(s+1)} + \frac{\frac{1}{2}s + \frac{11}{2}}{s^2+2s+5}$$

$$= \frac{-1}{2} \left( \frac{1}{s+1} \right) + \frac{1}{2} \frac{s+11}{(s+1)^2+4} = \frac{-1}{2} \left( \frac{1}{s+1} \right) + \frac{1}{2} \left[ \frac{(s+1)+10}{(s+1)^2+4} \right]$$



$$\begin{aligned}
 &= \frac{-1}{2} \left( \frac{1}{s+1} \right) + \frac{1}{2} \left[ \frac{s+1}{(s+1)^2+4} + \frac{10}{(s+1)^2+4} \right] \\
 \therefore L^{-1} \left[ \frac{5s+3}{(s+1)(s^2+2s+5)} \right] &= L^{-1} \left\{ \frac{1}{2} \left( \frac{1}{s+1} \right) + \frac{1}{2} \left[ \frac{s+1}{(s+1)^2+4} + \frac{10}{(s+1)^2+4} \right] \right\} \\
 &= e^{-t} L^{-1} \left\{ \frac{1}{2} \left( \frac{1}{5} \right) + \frac{1}{2} \left( \frac{s}{s^2+4} \right) + 5 \left( \frac{1}{s^2+4} \right) \right\} \\
 &= e^{-t} \left\{ \frac{1}{2} (1) \frac{1}{2} \cos(2t) + 5 \cdot \frac{1}{2} \sin 2t \right\} \\
 \therefore L^{-1} \left[ \frac{5s+3}{(s+1)(s^2+2s+5)} \right] &= \frac{1}{2} e^{-t} (1 + \cos 2t + 5 \sin 2t)
 \end{aligned}$$

Q.3(e) Find the Laplace transform of :  $f(t) = \begin{cases} 1 & 0 < t < a \\ -1 & a < t < 2a \end{cases}$  and  $f(t) = f(t+2a)$  [5]

Ans.:  $f(t) = f(t+2a)$

Above function is periodic function with period  $T = 2a$

$\therefore$  LT of periodic function is given as :

$$\begin{aligned}
 L[f(t)] &= \frac{1}{1-e^{-sT}} \int_0^T e^{-su} f(u) du \\
 \therefore L[f(t)] &= \frac{1}{1-e^{-2as}} \left[ \int_0^a e^{-su} (1) du + \int_a^{2a} e^{-su} (-1) du \right] = \frac{1}{1-e^{-2as}} \left\{ \left[ \frac{e^{-su}}{-s} \right]_0^a - \left[ \frac{e^{-su}}{-s} \right]_a^{2a} \right\} \\
 &= \frac{1}{1-e^{-2as}} \left\{ \left[ \frac{e^{-as}}{-s} - \frac{1}{-s} \right] - \left[ \frac{e^{-2as}}{-s} - \frac{e^{-as}}{-s} \right] \right\} \\
 &= \frac{1}{(-s)(1-e^{-2as})} [e^{-as} - 1 - e^{-2as} + e^{-as}] = \frac{-e^{-2as} + 2e^{-as} - 1}{(-s)(1-e^{-2as})} \\
 &= \frac{1 - 2e^{-as} + e^{-2as}}{s(1-e^{-2as})} = \frac{(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})} = \frac{1-e^{-as}}{s(1+e^{-as})} \\
 \therefore L[f(t)] &= \frac{1-e^{-as}}{s(1+e^{-as})}
 \end{aligned}$$

Q.3(d) Obtain the inverse Laplace transform of each of the given function  $\frac{(s+1)}{s^3(s-3)^2}$  [5]

Ans.: Let;  $\frac{s+1}{s^3(s-3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-3} + \frac{E}{(s-3)^2}$

$$(s+1) = As^2(s-3)^2 + Bs(s-3)^2 + c(s-3)^2 + Ds^3(s-3) + Es^3$$

Assuming values of S we get,

$$\therefore A = \frac{1}{9}; B = \frac{5}{27}; C = \frac{1}{9}; D = \frac{-1}{9}; E = \frac{4}{27}$$

$$\therefore \frac{1/9}{s} + \frac{5/27}{s^2} + \frac{1/9}{s^3} - \frac{1/9}{s-3} + \frac{4/27}{(s-3)^2}$$

$$\therefore \frac{1}{9} + \frac{5}{27}t + \frac{1}{18}t^2 - \frac{1}{9}e^{3t} + \frac{4}{27}te^{3t}$$

Q.4 Attempt any THREE of the following:

[15]

Q.4(a) Evaluate :  $\int_0^1 \int_0^1 \frac{dx dy}{\sqrt{(1-x^2)(1-y^2)}}$

[5]

$$\begin{aligned} \text{Ans.: Let } I &= \int_0^1 \int_0^1 \frac{dx dy}{\sqrt{(1-x^2)(1-y^2)}} = \int_0^1 \frac{1}{\sqrt{1-x^2}} dx \cdot \int_0^1 \frac{dy}{\sqrt{1-y^2}} \\ &= \int_0^1 \frac{1}{\sqrt{1-x^2}} dx [\sin^{-1} y]_0^1 = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \left[ \frac{\pi}{2} - 0 \right] = \frac{\pi}{2} \int_0^1 \frac{dx}{\sqrt{1-x^2}} \\ &= \frac{\pi}{2} [\sin^{-1} x]_0^1 = \frac{\pi}{2} \left[ \frac{\pi}{2} - 0 \right] = \frac{\pi^2}{4} \end{aligned}$$

Q.4(b) Evaluate  $\int_0^3 \int_0^{\sqrt{1+y^2}} \frac{dx dy}{(1+x^2+y^2)}$

[5]

$$\begin{aligned} \text{Ans.: } I &= \int_0^3 \int_0^{\sqrt{1+y^2}} \frac{dx dy}{(1+y^2)+x^2} \\ &= \int_0^3 \int_0^A \frac{dx dy}{(A^2+x^2)} \quad \text{putting } \sqrt{1+y^2} = A \text{ for convenience} \\ &= \int_0^3 \left[ \frac{1}{A} \tan^{-1} \frac{x}{A} \right]_0^A dy = \int_0^3 \frac{1}{A} \left[ \tan^{-1} \frac{A}{A} - \tan^{-1} 0 \right] dy \\ &= \int_0^3 \frac{1}{A} [\tan^{-1} 1 - 0] dy = \int_0^3 \frac{1}{A} \frac{\pi}{4} dy \\ \text{Now, } I &= \frac{\pi}{4} \int_0^3 \frac{1}{\sqrt{1+y^2}} dy = \frac{\pi}{4} [\log[y + \sqrt{1+y^2}]]_0^3 \\ &= \frac{\pi}{4} [\log(3 + \sqrt{10}) - \log 0] = \frac{\pi}{4} \log(3 + \sqrt{10}) \end{aligned}$$

Q.4(c) Evaluate  $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z} dx dy dz$

[5]

$$\begin{aligned} \text{Ans.: } &\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z} dx dy dz \\ &\text{Int. w.r.t. } z \\ &= \int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y} [e^z]_0^{x+\log y} dx dy \\ &= \int_0^{\log 2} \int_0^x e^{x+y} (e^{x+\log y} - e^0) dx dy = \int_0^{\log 2} \int_0^x (e^{2x} ye^y - e^x e^y) dx dy \\ &\text{Int. w.r.t. } y \\ &= \int_0^{\log 2} e^{2x} [ye^y - e^y]_0^x dx - \int_0^{\log 2} e^x [e^y]_0^x dx \\ &= \int_0^{\log 2} e^{2x} [(xe^x - e^x) - (0 - e^0)] dx - \int_0^{\log 2} e^x [e^x - e^0] dx \\ &= \int_0^{\log 2} (xe^{3x} - e^{3x} + 1) dx - \int_0^{\log 2} (e^{2x} - ex) dx \\ &= \left[ \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + x - \frac{e^{3x}}{3} - \frac{e^{-2x}}{2} + e^x \right]_0^{\log 2} \\ &= \log 2 \frac{e^{3\log 2}}{3} - \frac{e^{3\log 2}}{9} + \log 2 - \frac{e^{3\log 2}}{3} - \frac{e^{2\log 2}}{2} + e^{\log 2} \end{aligned}$$

$$\begin{aligned}
 &= \frac{8}{3} \log 2 - \frac{8}{9} + \log 2 - \frac{8}{3} - \frac{4}{2} + 2 \\
 &= \log 2 \left( \frac{8}{3} + 1 \right) - \frac{8}{9} - \frac{8}{3} \\
 &= \frac{11}{3} \log 2 - \frac{32}{9}
 \end{aligned}$$

Q.4(d) Change the order of integration and evaluate  $\int_0^2 \int_0^{x^2/4} xy \, dx \, dz$  [5]

Ans.: Here  $y = \frac{x^2}{4}$  and  $y = 0$

$$x = 0 \text{ to } x = 2$$

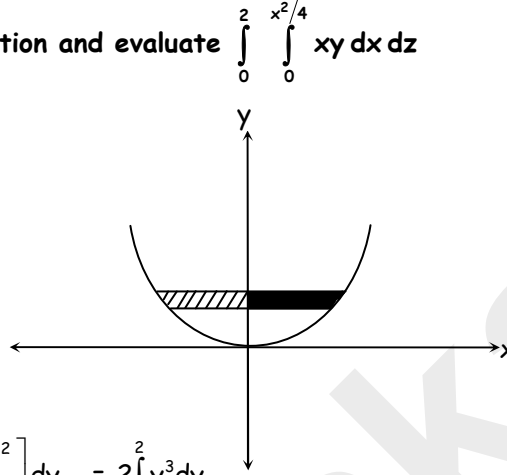
$$x^2 = 4y \text{ and } y = 0$$

After change of order

$$= \int_0^2 \int_0^{2y} xy \, dx \, dy$$

Int. w.r.t. x

$$\begin{aligned}
 &= \int_0^2 y \left[ \frac{x^2}{2} \right]_0^{2y} dy &= \int_0^2 y \left[ \frac{4y^2}{2} \right] dy &= 2 \int_0^2 y^3 dy \\
 &= 2 \frac{[y^4]_0^2}{4} &= \frac{2 \times 16}{4} &= 8
 \end{aligned}$$



Q.4(e) Evaluate  $\iint y \, dx \, dy$  over the area bounded by  $y = x^2$ ,  $x + y = 2$  [5]

Ans.: Area bounded by  $y = x^2$  (parabola) &  $x + y = 2$

The point of intersection of  $y = x^2$  &  $x + y = 2$

$$\therefore x + x^2 = 2$$

$$\therefore x^2 + x - 2 = 0$$

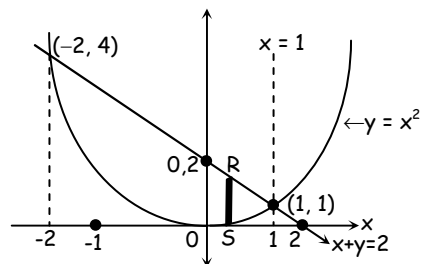
$$\therefore x = 1, -2$$

At  $x = 1$ ;  $y = 1$  & at  $x = -2$ ;  $y = 4$

$\therefore (1, 1)$  is the point of intersection in 1<sup>st</sup> quadrant. Take a vertical strip SR; along SR, x constant & y varies from to R i.e.  $y = x^2$  to  $y = 2 - x$

Now slide strip SR keeping 2<sup>nd</sup> quadrant to y - axis therefore y constant & x varies from  $x = 0$  to  $x = 1$

$$\begin{aligned}
 \text{Step 2: I} &= \int_0^1 \int_{x^2}^{2-x} y \, dx \, dy &= \int_0^1 dx \left[ \int_{x^2}^{2-x} y \, dy \right] \\
 &= \int_0^1 dx \cdot \frac{[y^2]^{2-x}}{2} &= \frac{1}{2} \int_0^1 [(2-x)^2 - x^4] \, dx \\
 &= \frac{1}{2} \int_0^1 (4 - 4x + x^2 - x^4) \, dx \\
 &= \frac{1}{2} \left[ 4x - 2x^2 + \frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 \\
 &= \frac{1}{2} \left[ 4 - 2 + \frac{1}{3} - \frac{1}{5} \right] &= \frac{1}{2} \times \frac{32}{15}
 \end{aligned}$$



$$I = \frac{16}{15}$$

Q.4(f) Evaluate  $\int_0^3 \int_0^{\sqrt{1+y^2}} \frac{dx dy}{(1+x^2+y^2)}$  [5]

Ans.: 
$$I = \int_0^3 \int_0^{\sqrt{1+y^2}} \frac{dx dy}{(1+y^2)+x^2}$$

$$= \int_0^3 \int_0^A \frac{dx dy}{(A^2+x^2)} \quad \text{putting } \sqrt{1+y^2} = A \text{ for convenience}$$

$$= \int_0^3 \left[ \frac{1}{A} \tan^{-1} \frac{x}{A} \right]_0^A dy = \int_0^3 \frac{1}{A} \left[ \tan^{-1} \frac{A}{A} - \tan^{-1} 0 \right] dy$$

$$= \int_0^3 \frac{1}{A} [\tan^{-1} 1 - 0] dy = \int_0^3 \frac{1}{A} \frac{\pi}{4} dy$$

Now, 
$$I = \frac{\pi}{4} \int_0^3 \frac{1}{\sqrt{1+y^2}} dy = \frac{\pi}{4} [\log[y + \sqrt{1+y^2}]]_0^3$$

$$= \frac{\pi}{4} [\log(3 + \sqrt{10}) - \log 0] = \frac{\pi}{4} \log(3 + \sqrt{10})$$

Q.5 Attempt the following (any THREE) [15]

Q.5(c) Evaluate  $\int_0^{2a} x(2ax - x^2)^{1/2} dx$  [5]

Ans.: 
$$\int_0^{2a} x(2ax - x^2)^{1/2} dx = \int_0^{2a} x^{3/2} (2a - x^2)^{1/2} dx$$

Let  $x = 2a \sin^2 \theta$   
 $\therefore dx = 4a \sin \theta \cos \theta d\theta$   
 Where  $x = 2a$ ; .....  $\theta \rightarrow \pi/2$   
 $x = 0$ ; .....  $\theta \rightarrow 0$

$$= \int_0^{\pi/2} (2a \sin^2 \theta)^{3/2} (2a - 2a \sin^2 \theta)^{1/2} \times 4a \sin \theta \cos \theta d\theta$$

$$= 16a^3 \int_0^{\pi/2} \sin^3 \theta \cos \theta \sin \theta \cos \theta d\theta = 16a^3 \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta d\theta$$

$$= \frac{16a^3 \left[ \frac{5}{2} \frac{3}{2} \right]}{14} = \frac{16a^3 \times \frac{3}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}}{2 \times 16}$$

$$= \frac{16a^3 \times 3 \times \pi}{2 \times 16} = \frac{16a^3 \times 3 \times \pi}{16 \times 6} = \frac{\pi a^3}{2}$$

Q.5(b) Evaluate  $\int_0^{\pi/2} \sin^6 x \cos^7 x dx$  [5]

Ans.: Where  $R = 6$  and  $q = 7$

$$\frac{\frac{R+1}{2} \frac{q+1}{2}}{2 \frac{R+q+2}{2}} = \frac{\frac{7}{2} \frac{4}{2}}{2 \frac{15}{2}}$$

$$= \frac{\frac{7}{2} \times 3}{2 \times \frac{13}{2} \times \frac{11}{2} \times \frac{9}{2} \times \frac{7}{2} \times \frac{7}{2}}$$

$$= \frac{8 \times 6}{13 \times 11 \times 9 \times 7} = \frac{16}{3003}$$

Q.5(c) Show that  $\int_0^1 \frac{x^a - x^b}{\log x} \log \left( \frac{a+1}{b+1} \right)$  using DUIS. [5]

Ans.: "a" is a parameter & "b" is constant

Step 1:  $I(a) = \int_0^1 \frac{x^a - x^b}{\log x} dx \quad \dots (1)$

Applying DUIS

$$I'(a) = \int_0^1 \frac{1}{\log x} \left[ \frac{\partial}{\partial a} (x^a - x^b) \right] dx \Rightarrow \left( \frac{\partial}{\partial a} x^b = 0 \text{ as } x^b \text{ is constant} \right)$$

$$I'(a) = \int_0^1 \frac{1}{\log x} x^a \log x dx = \int_0^1 x^a dx = \left[ \frac{x^{a+1}}{a+1} \right]_0^1 = \frac{1}{a+1}$$

Step 2:  $\therefore I(a) = \int \frac{1}{a+1} da + A$

$\therefore I(a) = \log(a+1) + A \quad \dots (2)$

Step 3: Put  $a = b$  in equation (1)

$$I(b) = 0$$

Put  $a = b$  in equation (2)

$$I(b) = \log(b+1) + A$$

$$\therefore A = -\log(b+1)$$

$\therefore$  From equation (2),

$$I(a) = \log(a+1) - \log(b+1)$$

$$= \log \left( \frac{a+1}{b+1} \right) \quad \dots \text{Proved}$$

Q.5(d) If  $y = \int_0^x f(t) \sin[a(x-t)] \cdot dt$  then show that,  $\frac{d^2y}{dx^2} + a^2y = af(x)$  [5]

Ans.:  $\because x$  is a parameter

$$\therefore y = \int_0^x f(t) \sin[a(x-t)] dt$$

Applying Leibnitz rule of DUIS w.r.t "x"

$$\frac{dy}{dx} = \int_0^x \frac{\partial}{\partial x} \{f(t) \sin[a(x-t)]\} dt + f(x) \sin a(x-x) \frac{d}{dx} x - f(0) \sin(ax) \frac{d}{dx} 0$$

$$= \int_0^x f(t) a \cos[a(x-t)] dt + 0 - 0$$

$$\frac{dy}{dx} = a \int_0^x f(t) \cos[a(x-t)] dt$$

Again applying Leibnitz rule w.r.t. x

$$\frac{d^2y}{dx^2} = a \left[ \int_0^x \frac{\partial}{\partial x} f(t) \cos[a(x-t)] dt + f(x) \cos[a(x-x)] \frac{d(x)}{dx} - 0 \right]$$

$$= a \left[ -\int_0^x f(t) \sin a(x-t) a dt + f(x) \right] = -a^2 \int_0^x f(t) \sin a(x-t) dt + a f(x)$$

$$\frac{d^2y}{dx^2} = -a^2y + a f(x) \quad \left[ \because y = \int_0^x f(t) \sin a(x-t) dt \right]$$

$$\therefore \frac{d^2y}{dx^2} + a^2y = a f(x) \quad \dots \text{Proved}$$

**Q.5(e) Evaluate : (i)  $\operatorname{erfc}(-x) + \operatorname{erfc}(x)$  (ii)  $\operatorname{erfc}(x) + \operatorname{erfc}(x)$  [5]**

**Ans.:** (i)  $\operatorname{erfc}(-x) + \operatorname{erfc}(x)$

w.k.t

$$\operatorname{erfc}(x) + \operatorname{erfc}(x) = 1$$

$$\therefore \operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$

$$\therefore \operatorname{erfc}(x) + 1 - \operatorname{erf}(x)$$

w.k.t.

$$\operatorname{erf}(-x) = -\operatorname{erf}(x)$$

$$\therefore \operatorname{erfc}(-x) + 1 + \operatorname{erf}(-x) = 2$$

**(ii)  $\operatorname{erf}(x) + \operatorname{erfc}(x)$**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx \quad \text{and} \quad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-x^2} dx$$

$$\begin{aligned} \text{Now, } \operatorname{erf}(x) + \operatorname{erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx + \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-x^2} dx \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx = \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} = 1 \end{aligned}$$

**Q.5(e) Evaluate  $\int_0^1 \frac{x^7}{(1-x^4)^{1/2}} dx$  [5]**

**Ans.:**  $\int_0^1 \frac{x^7}{(1-x^4)^{1/2}} dx$

Let  $x^4 = t$ ; when  $x = 1$ ;  $t = 1$

$4x^3 dx = dt$  when  $x = 0$ ;  $t = 0$

$$= \frac{1}{4} \int_0^1 \frac{t dt}{(1-t)^{1/2}} = \frac{1}{4} \int_0^1 t^{2-1} (1-t)^{1/2-1} dt$$

$$= \frac{1}{4} \frac{2 \sqrt{1/2}}{\sqrt{5/2}} = \frac{1}{4} \frac{1 \times \sqrt{1/2}}{3/2 \times 1/2 \times \sqrt{1/2}}$$

$$= \frac{1}{3}$$

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